EE16B, Spring 2018
UC Berkeley EECS

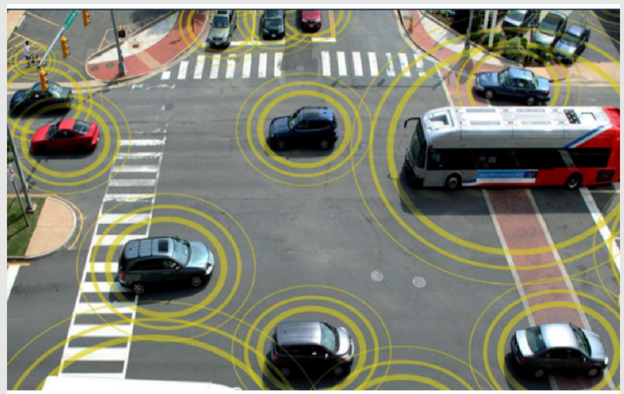
Maharbiz and Roychowdhury

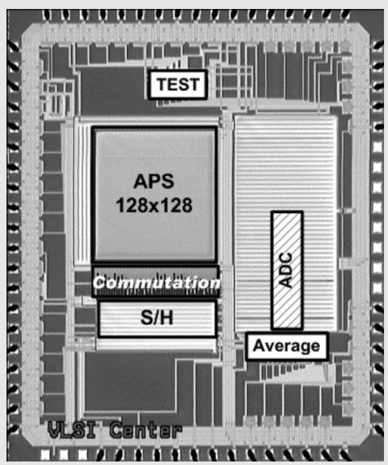
Lecture 4A: Overview Slides

State Space Representations

Systems

- Previously: circuits
- Now: systems
  - circuits + more: a broader concept

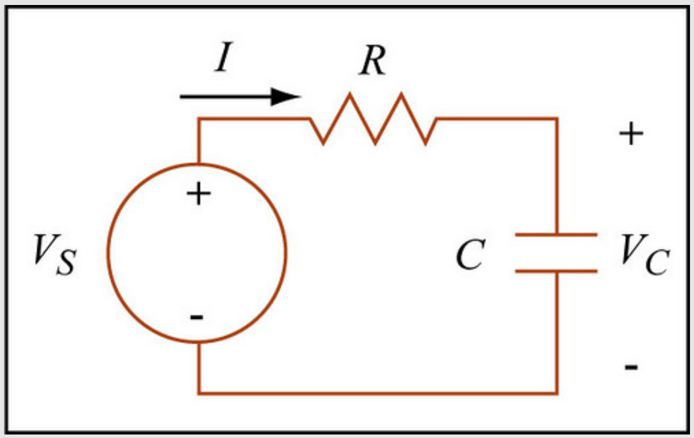


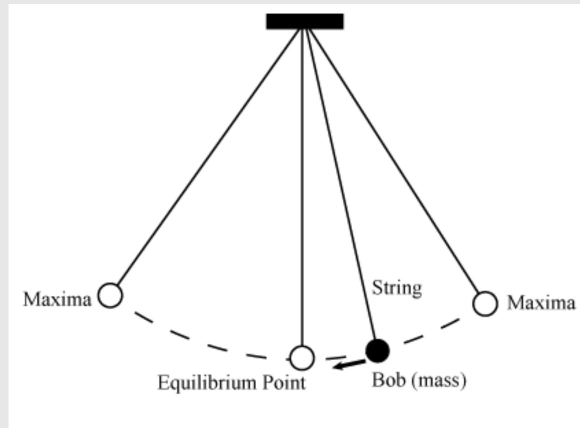


- Can be enormously complex
  - multi-domain
    - ➔ EE (control, comm., computing, ...)
    - ➔ mech., chem., optical, …
  - hierarchy of sub-systems

Simpler Systems

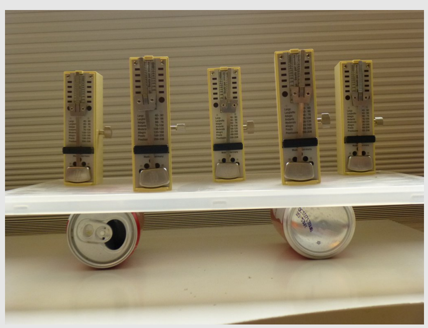
- … easier to understand and to work with





$$4H + O_2 \underset{k_2}{\overset{k_1}{\rightleftharpoons}} H_2O$$

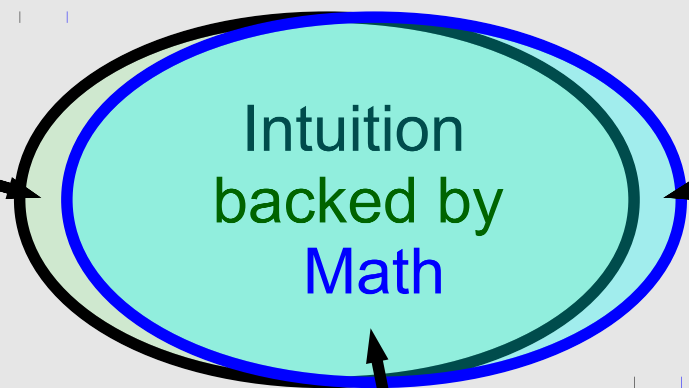
- Even small and simple systems can do interesting things



A Note about Intuition vs Math

- Proper intuition: extremely important
  - quickly make mental leaps to solve problems and design new things
    - … avoiding confusion that can arise from masses of detail
- **Intuition doesn't arise magically**
  - mathematics: essential for **in-depth** understanding
    - math + experience and practice → intuition

may be wrong!

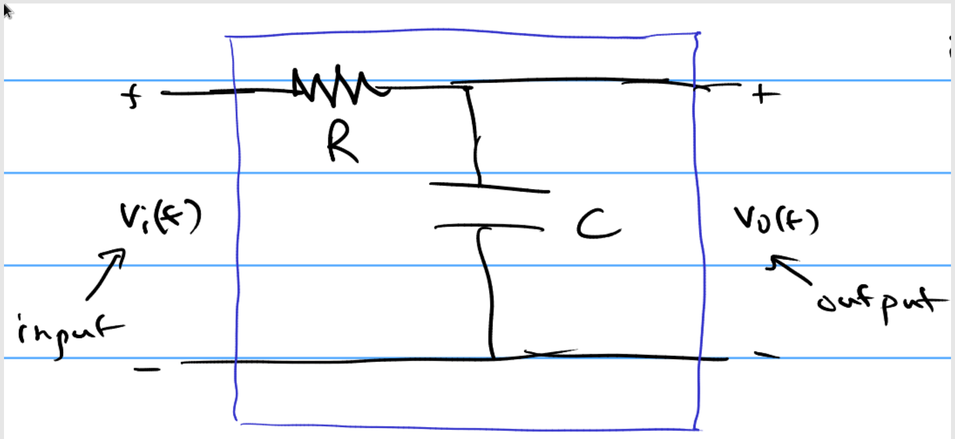


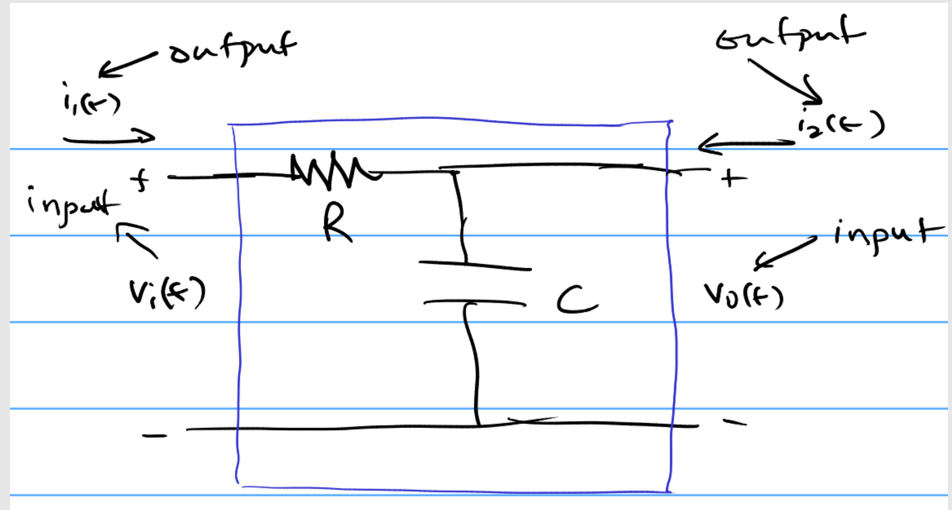
sometimes hard to translate math to intuition → not very useful for design
- but may be useful for, eg, understanding and computation

BEST: both!

Inputs and Outputs as **VECTORS**

- (move to xournal)





$$\vec{u}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \; ; \; \vec{y}(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

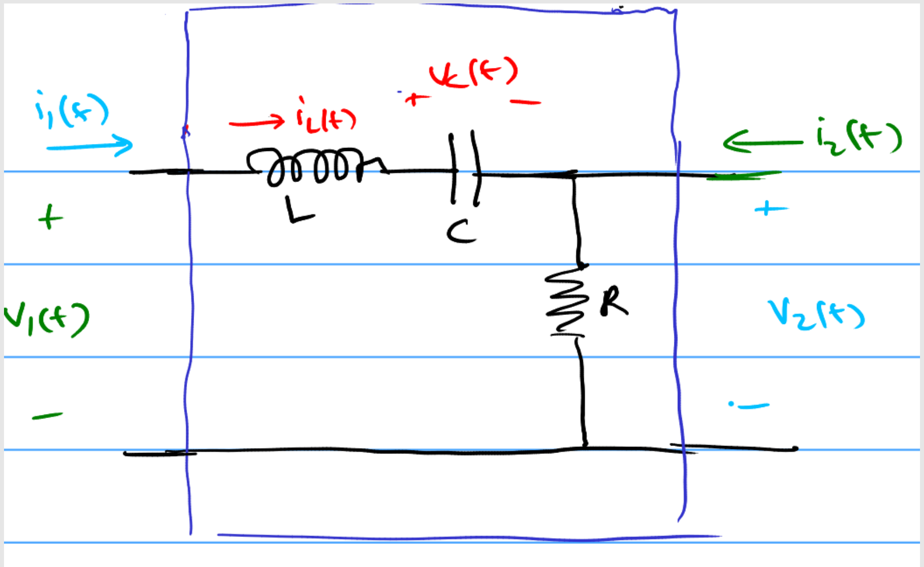
vector of inputs

vector of outputs

INPUTS AND OUTPUTS CAN BE ORGANIZED AS VECTORS

The Internal State

- (move to xournal)



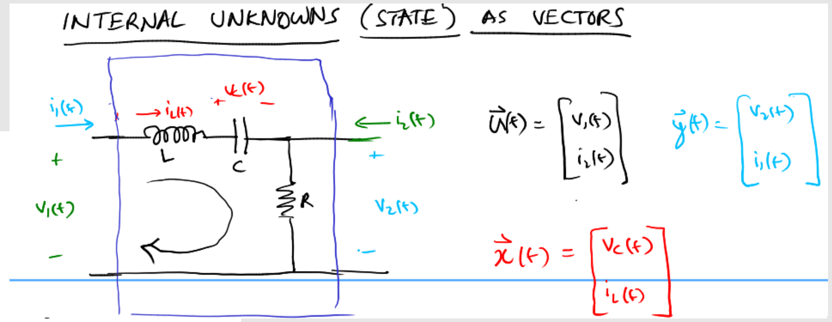
$$\vec{u}(t) = \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix} \qquad \vec{y}(t) = \begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix}$$

- Internal voltages/currents (unknowns): the **state**
  - also written as a vector: $\vec{x}(t)$

The System's Equations

- (move to xournal)



EQUATIONS

capacitor + KCL : $C\frac{dv_C(t)}{dt} - i_L(t) = 0$

KVL : $L\frac{di_L(t)}{dt} + v_C(t) + R\left(i_L(t) + i_2(t)\right) - v_1(t) = 0$

$$\underbrace{\begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}}_{A} \overbrace{\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}^{d/dt\,\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & R \end{bmatrix}}_{B} \overset{\vec{x}(t)}{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & R \end{bmatrix}}_{C} \overset{\vec{u}(t)}{\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}} = \overset{\vec{0}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} & 1/C \\ -1/L & -R/L \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} & \\ 1/L & -R/L \end{bmatrix}\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

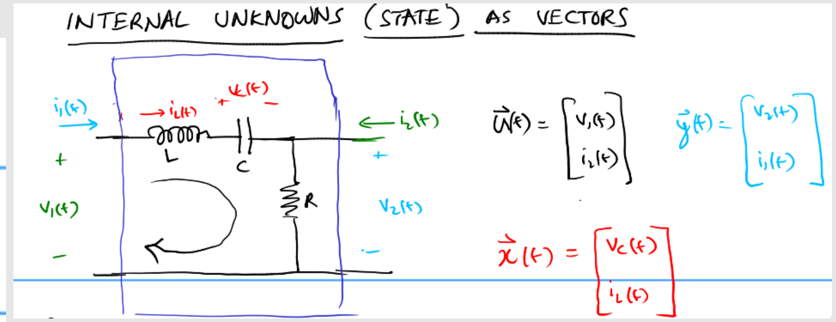
- can be written using the **input** and **state vectors**

The Outputs

- (move to xournal)

$$v_2(t) = R\left(i_L(t) + i_2(t)\right)$$

$$i_1(t) = i_2(t)$$



$$\underbrace{\begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix}}_{\vec{y}(t)} = \underbrace{\left[\begin{array}{c|c} 0 & R \\ \hline 0 & 1 \end{array}\right]}_{D} \underbrace{\begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\left[\begin{array}{c|c} 0 & R \\ \hline 0 & 0 \end{array}\right]}_{E} \underbrace{\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}}_{\vec{u}(t)}$$

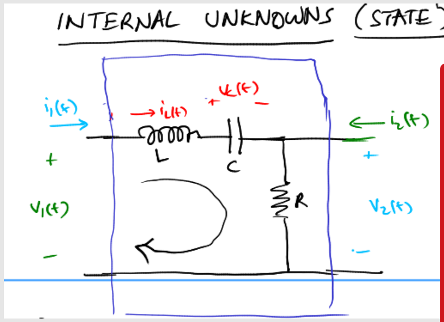
- also expressed using the **input** and **state vectors**

State+Output Eqns Together

$$\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t)\end{bmatrix} = \begin{bmatrix} & 1/C \\ -1/L & -R/L \end{bmatrix}\begin{bmatrix} v_C(t)\\ i_L(t)\end{bmatrix} + \begin{bmatrix} & \\ 1/L & -R/L\end{bmatrix}\begin{bmatrix} v_1(t)\\ i_2(t)\end{bmatrix}$$

$\vec{f}(\vec{x}, \vec{u})$ $\quad$ $\vec{g}(\vec{x}, \vec{u})$

INTERNAL UNKNOWNS (STATE) AS VECTORS



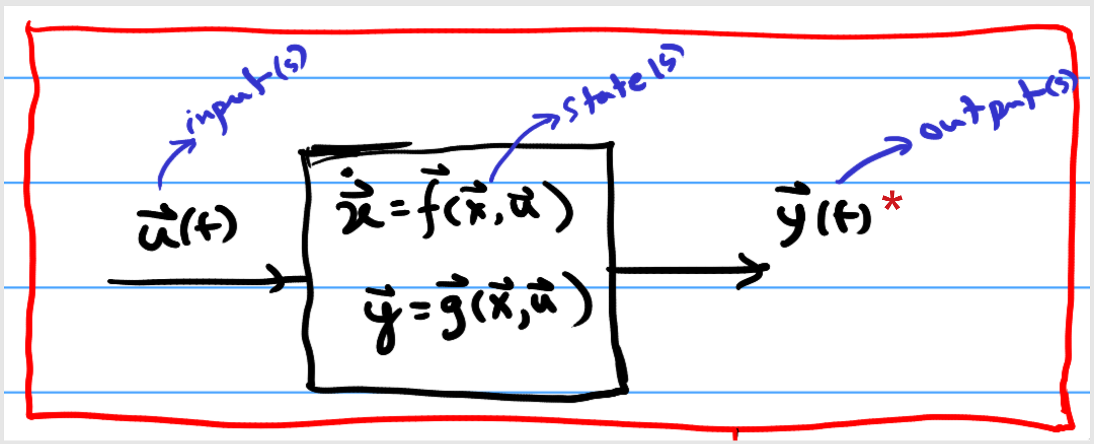
$\vec{u}(t) = \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$ $\quad$ $\vec{y}(t) = \begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix}$

$\vec{x}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$

$$\overset{\vec{y}(t)}{\begin{bmatrix} v_2(t) \\ i_1(t)\end{bmatrix}} = \underbrace{\begin{bmatrix} 0 & R \\ 0 & 1 \end{bmatrix}}_{D}\overset{\vec{x}(t)}{\begin{bmatrix} v_C(t)\\ i_L(t)\end{bmatrix}} + \underbrace{\begin{bmatrix} 0 & R \\ 0 & 0\end{bmatrix}}_{E}\overset{\vec{u}(t)}{\begin{bmatrix} v_1(t)\\ i_2(t)\end{bmatrix}}$$

- **general form:** $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t))$, $\quad \vec{y}(t) = \vec{g}(\vec{x}(t), \vec{u}(t))$

  \+ initial condition (IC)

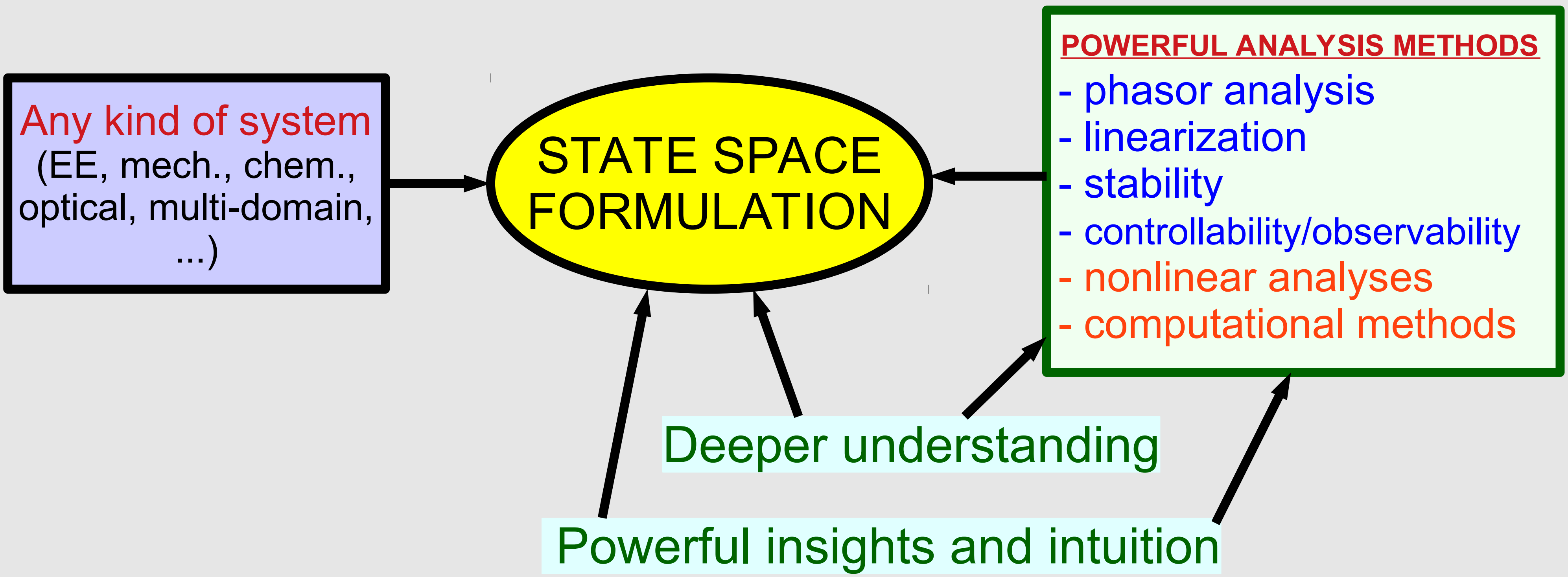


STATE SPACE FORMULATION

* if not explicitly given, the entire state will be the output.

State Space Formulation: Benefits

- why is it useful?
  - any circuit can be written like this (not just this one)*
    - however big or complicated
  - not just circuits – any system* from any domain!
    - including multi-domain systems



* some additional generalization needed – won't cover in this class

Previous Lecture

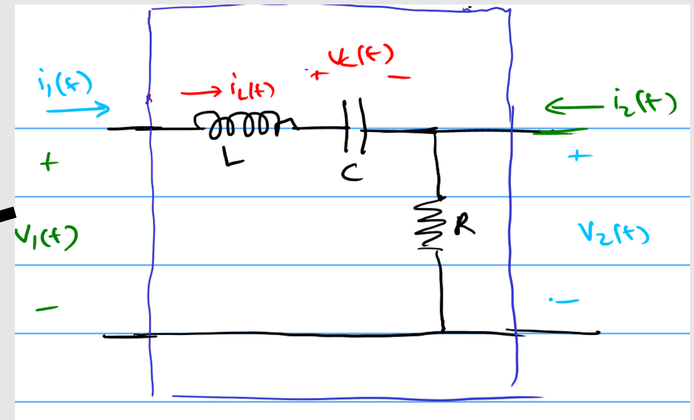
$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)) \qquad \vec{y}(t) = \vec{g}(\vec{x}(t), \vec{u}(t))$$

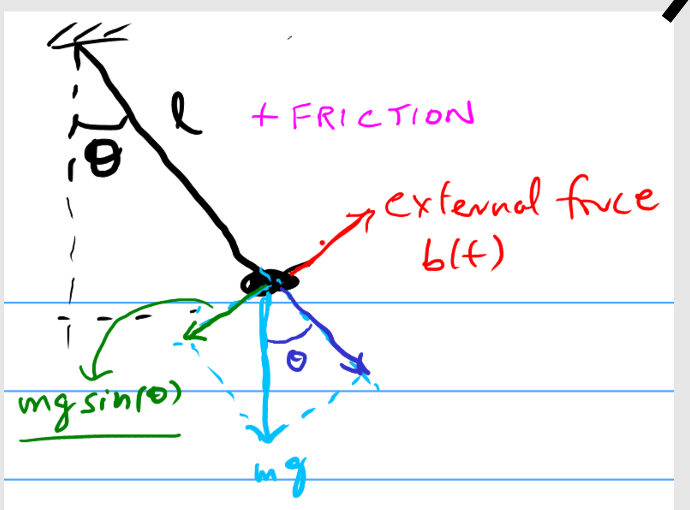
+ initial condition (IC)

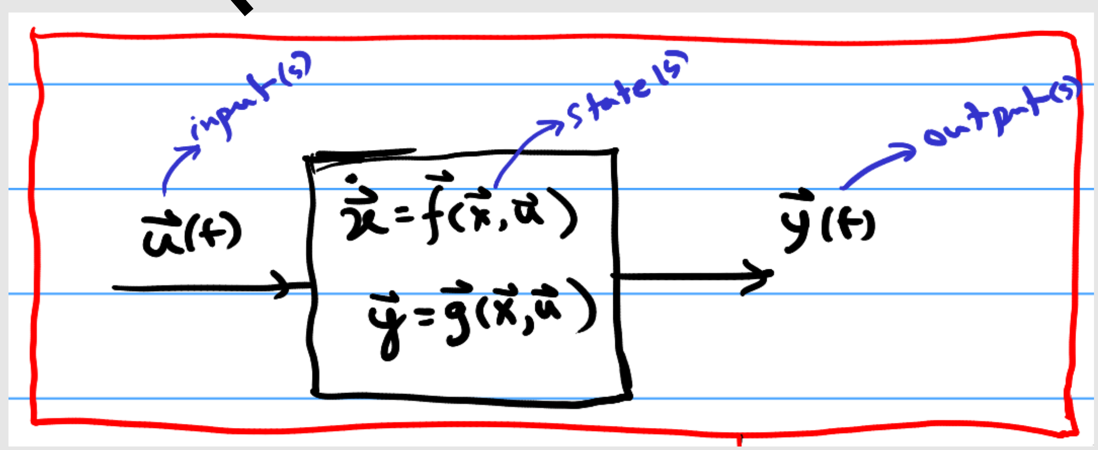
- examples
  - RLC circuit
  - pendulum

Any kind of system (EE, mech., chem., optical, multi-domain, ...)

STATE SPACE FORMULATION

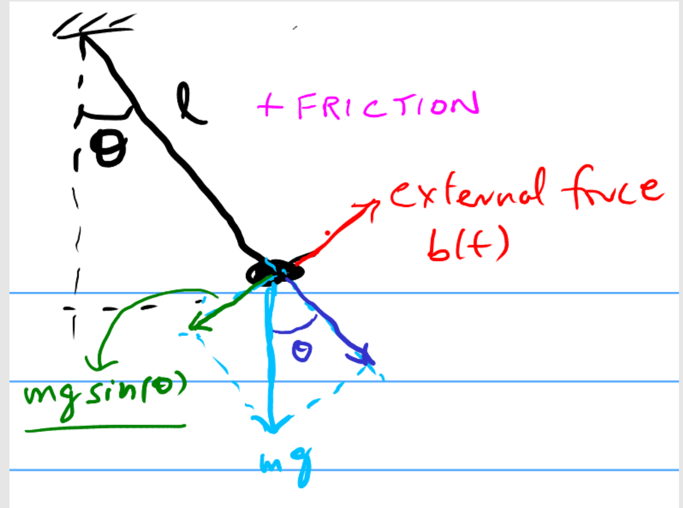






Mechanical Example: Pendulum

- (move to xournal)



→ Newton's eqn. of motion: $F = ma$ or $a = \frac{F}{m}$

→ total tangential force =

force due to gravity : $-mg\sin(\theta)$

\+ force " " friction : $-k \cdot$ velocity

\+ externally applied force : $b(t)$

→ arc-length (from bottom) $= y = \ell\theta$ ← θ in radians

→ velocity $= \frac{dy}{dt} = \ell \underbrace{d\theta/dt}_{v_\theta}$

→ acceleration $= \frac{d^2y}{dt^2} = \ell\frac{d^2\theta}{dt^2} = \ell\frac{dv_\theta}{dt}$

→ total force : $-mg\sin(\theta) - k\ell\frac{d\theta}{dt} + b(t)$

→ $a = F/m \implies \ell\frac{d^2\theta}{dt^2} = -g\sin(\theta) - \frac{k\ell}{m}\frac{d\theta}{dt} + \frac{b(t)}{m}$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin(\theta) - \frac{k}{m}\frac{d\theta}{dt} + \frac{b(t)}{m\ell}}$$

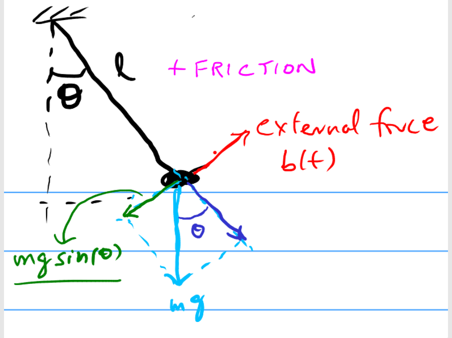
$$\vec{x} = \begin{bmatrix}\theta \\ v_\theta\end{bmatrix}, \quad \vec{u} = [b(t)]$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -\frac{g}{\ell}\sin(\theta) - \frac{k}{m}v_\theta + \frac{b(t)}{m\ell}\end{bmatrix}$$

- The system is nonlinear (because of sin(...))

Pendulum: simplification for small θ

- $$\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -\frac{g}{\ell}\sin(\theta) - \frac{k}{m} v_\theta + \frac{b(t)}{m\ell} \end{bmatrix}$$



− if θ is small (check in MATLAB)

→ $\sin(\theta) \simeq \theta$ (in radians)

→ example of linearization

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -g/\ell & -\frac{k}{m} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m\ell} \end{bmatrix} \vec{u}(t)$$

- Does this look familiar?

$$\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

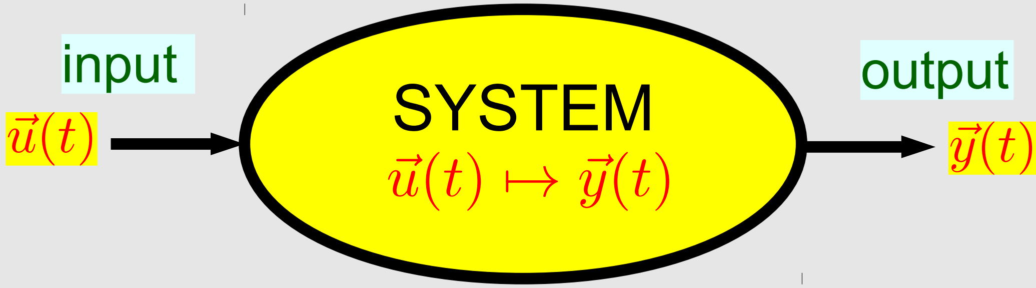
- **INSIGHT**: RLC ckt and damped pendulum are “the same”!

Filters

Filters?? → MEMS filters! (smaller, better, cheaper)

Recap of Linearity

- The concept of LINEARITY is extremely important
  - **it is fundamentally a systems concept**



- 3 steps to check linearity
  - **clearly identify your inputs and outputs (this affects linearity)**
  - **check superposition**
    - if $\vec{u}_1(t) \mapsto \vec{y}_1(t)$ and $\vec{u}_2(t) \mapsto \vec{y}_2(t)$, then $(\vec{u}_1(t) + \vec{u}_2(t)) \mapsto (\vec{y}_1(t) + \vec{y}_2(t))$, $\forall \vec{u}_1(t), \vec{u}_2(t)$
  - **check scaling**
    - if $\vec{u}(t) \mapsto \vec{y}(t)$, then $(\alpha \vec{u}(t)) \mapsto (\alpha \vec{y}(t))$, $\forall \vec{u}(t)$ and $\forall \alpha \in \Re$

Linearity of State Space Formulations

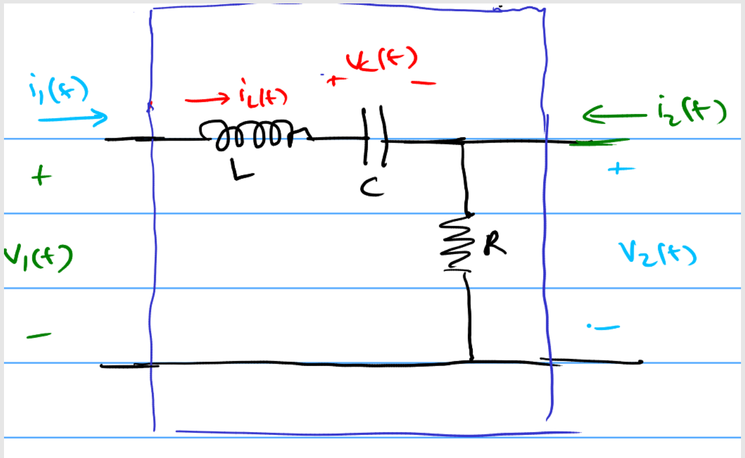
- general S.S.F: $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t))$, $\vec{y}(t) = \vec{g}(\vec{x}(t), \vec{u}(t))$
- If $\vec{f}(\vec{x}(t), \vec{u}(t)) \equiv A\vec{x}(t) + B\vec{u}(t)$ and $\vec{g}(\vec{x}(t), \vec{u}(t)) \equiv C\vec{x}(t) + D\vec{u}(t)$

matrices

matrices

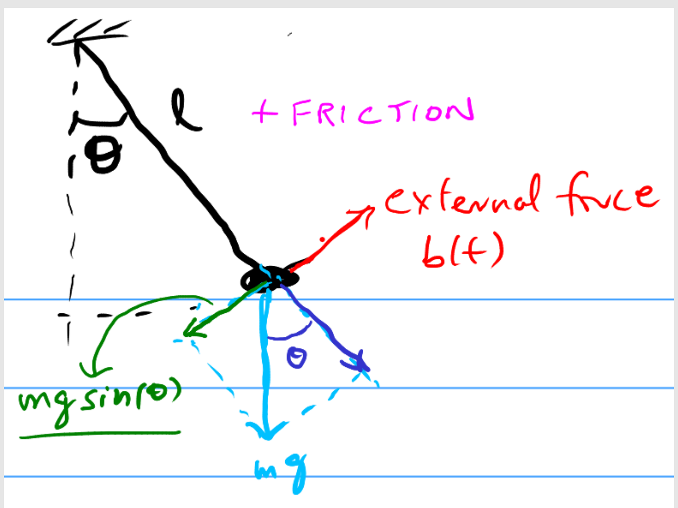
  - **then the system is linear**
- Proof?

Are these Systems Linear?



$$\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} & 1/C \\ -1/L & -R/L \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} & \\ 1/L & -R/L \end{bmatrix}\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} v_2(t) \\ i_1(t) \end{bmatrix}}_{\vec{y}(t)} = \underbrace{\begin{bmatrix} 0 & R \\ 0 & 1 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}}_{\vec{u}(t)}$$



$$\vec{x} = \begin{bmatrix} \theta \\ v_\theta \end{bmatrix}, \quad \vec{u} = [b(t)]$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -g/l \sin(\theta) - k/m\, v_\theta + \frac{b(t)}{ml} \end{bmatrix}$$

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} v_\theta \\ -\frac{g}{l}\theta - \frac{k}{m}v_\theta + \frac{b(t)}{ml} \end{bmatrix}$$

State Space F. for Discrete Time Systems

- What is a discrete-time system?
  - example: compound interest
    - ➔ (move to xournal)

— Example: compound interest

— Principal P

— Annual rate of interest r, compounded monthly ← DISCRETE (integer, not real number)

— Savings $S[t] = S[t-1] + \frac{r}{12} S[t-1]$, $t \to 1, 2, 3, \ldots$

with $S[0] = P$ ← INITIAL CONDITION

— Additions/withdrawals each month: $u[t]$

— $\underbrace{S[t] = S[t-1] + \frac{r}{12} S[t-1] + u[t]}_{\text{DISCRETE TIME}}$, $t \to 1, 2, 3$

— with $S[0] = P$

- **general form:** $\vec{x}[t+1] = \vec{f}(\vec{x}[t], \vec{u}[t])$, $\vec{y}[t] = \vec{g}(\vec{x}[t], \vec{u}[t])$

  $t = 1, 2, 3, \cdots$ + initial condition (IC) $\vec{x}[0]$ given

STATE SPACE FORMULATION

- Discrete time sometimes more natural (eg, for finance, social dynamics, ...)

Another D.T. Example: Profs. and PhDs.

- (move to xournal)

– $p[t]$: no. of profs. in the US, year t $(t = 1, 2, 3, \ldots)$

– $r[t]$: no. of PhDs in year t

– γ : fraction of PhDs who become professors

– δ : fraction in each profession retiring

– $u[t]$: average number of PhD students graduated per prof. per year

can be manipulated by the professor (controlled by, eg, funding)

– Q: how do $p(t)$ and $r(t)$ evolve with time?

$$p[t+1] = p[t] - \delta p[t] + \gamma r[t]$$

$$r[t+1] = r[t] - \delta r[t] - \gamma r[t] + p[t] u[t]$$

– State space repr. ?

$$\vec{x}(t) \triangleq \begin{bmatrix} p[t] \\ r[t] \end{bmatrix}; \quad \vec{f}(\vec{x}) = \begin{bmatrix} p(1-\delta) + \gamma r \\ r(1-\delta-\gamma) + pu \end{bmatrix}$$

VECTOR D.T. STATE-SPACE FORMULATION

- Linear?