

NOTES FOR LECTURES 6B & 7A: CONTROLLER CANONICAL FORM & OBSERVABILITYJAITEET ROYCHOWDHURYCONTROLLER CANONICAL FORM (CCF)

→ Recall a previous ^{controllability} example: $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$

→ its characteristic polynomial $c(\lambda) \triangleq \det(A - \lambda I)$ had a nice form:

$$\rightarrow -\lambda(a_2 - \lambda) - a_1 \equiv \boxed{\lambda^2 - a_2\lambda - a_1}$$

→ A rather elegant observation someone made long ago is this:

→ suppose you have a size n matrix in the same form:

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 0 & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}; \text{ and } \vec{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

TOGETHER CALLED CONTROLLER CANONICAL FORM (CCF)

→ then, $\boxed{c(\lambda) \triangleq \det(A - \lambda I) = \lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \dots - a_3\lambda^3 - a_2\lambda^2 - a_1\lambda}$

→ proof: easily (though laboriously) obtained by applying the formula for determinants involving expansion using minors to $(A - \lambda I)$. Try it yourself if you have some time on your hands and you like getting to the bottom of things.

→ Now, suppose you have a scalar input to your system $u(t)$ and

the system is: $\underbrace{\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}u(t)}$

this (the original system without feedback) is called the OPEN LOOP system

→ Now, if you apply feedback $u(t) \mapsto u(t) - \vec{k}^T \vec{x}(t)$, then the "closed loop system" becomes:

$$\dot{\vec{x}} = (A - \vec{b} \vec{k}^T) \vec{x} + \vec{b} u(t),$$

with $(A - \vec{b} \vec{k}^T) =$

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ a_1 - k_1 & a_2 - k_2 & a_3 - k_3 & \dots & a_n - k_n \end{bmatrix}$$

→ with characteristic polynomial: $C_f(\lambda) \triangleq \det(A - \vec{b} \vec{k}^T - \lambda I)$

$$= \lambda^n - (a_n - k_n) \lambda^{n-1} - (a_{n-1} - k_{n-1}) \lambda^{n-2} - \dots - (a_3 - k_3) \lambda^2 - (a_2 - k_2) \lambda - (a_1 - k_1)$$

→ Now, our goal is to choose $k_1, \dots, k_n$ (the feedback) to place the eigenvalues of $(A - \vec{b} \vec{k}^T)$, i.e., the roots of $C_f(\lambda)$, wherever we want.

→ Suppose we want the roots to be $\lambda_1, \lambda_2, \dots, \lambda_n$

↳ remember that if complex, the conjugate must also be present.

→ i.e., the desired char. poly. is:

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \dots (\lambda - \lambda_n)$$

$$= \lambda^n - \underbrace{(\lambda_1 + \lambda_2 + \dots + \lambda_n)}_{b_n} \lambda^{n-1} + \underbrace{\left(\begin{matrix} \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \dots + \lambda_1 \lambda_n \\ + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \dots + \lambda_2 \lambda_n \\ + \lambda_3 \lambda_4 + \dots + \lambda_3 \lambda_n \\ + \lambda_{n-1} \lambda_n \end{matrix} \right)}_{b_{n-1}} \lambda^{n-2} + \dots + (-1)^n \underbrace{\lambda_1 \lambda_2 \lambda_3 \dots \lambda_n}_{b_1}$$

→ i.e., $b_1, \dots, b_n$ can be calculated from $\lambda_1, \dots, \lambda_n$, though the process may be extremely tedious.

→ now all you have to do to devise the feedback is equate the coefficients of the 2 char. poly. expressions:

$$a_n - k_n = -b_n$$

$$a_{n-1} - k_{n-1} = -b_{n-1}$$

⋮

$$a_1 - k_1 = -b_1$$

→ I.e, if $(A, \vec{b})$ is in controllability canonical form, then it is always possible to devise feedback to place the eigenvalues wherever you like.

→ Size 3 example:

$$\rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}; \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

↓

$$\rightarrow A - \vec{b} \vec{k}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-k_1 & 2-k_2 & 3-k_3 \end{bmatrix}$$

$$\rightarrow \text{char. poly.}: \lambda^3 - (3-k_3)\lambda^2 - (2-k_2)\lambda - (1-k_1)$$

→ Suppose we want the 3 new eigenvalues to be:

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$$

$$\Rightarrow (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3$$

→ Therefore we should set: $k_3 = 3, k_2 = 2, k_1 = 1$

→ Suppose we want $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$

$$\Rightarrow (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \lambda(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) - \lambda_1\lambda_2\lambda_3$$

$$= \lambda^3 + 6\lambda^2 + \lambda(2+6+3) + 6$$

$$\Rightarrow -(3 - k_3) = -6 \Rightarrow k_3 = -3$$

$$\Rightarrow -(2 - k_2) = -11 \Rightarrow k_2 = -9$$

$$\Rightarrow -(1 - k_1) = -6 \Rightarrow k_1 = -5$$

→ But can any system be put in controller canonical form?

→ A: yes, if it is controllable!

→ Here is how you put any controllable system in c.c.f.:

1. Form the controllability matrix:

$$R_n \triangleq [\vec{b}, A\vec{b}, A^2\vec{b}, \dots, A^{n-1}\vec{b}] \rightarrow \text{a square matrix}$$

— if the system is controllable, then R_n is full rank, i.e., invertible.

2. Calculate R_n^{-1}

3. Take the last row of R_n^{-1} — call this last row $\vec{q}_V^T$ ($\vec{q}_V$ is a col. vector; $\vec{q}_V^T$ a row vector)

4. Form the following matrix, row by row:

$$T \triangleq \begin{bmatrix} \leftarrow \vec{q}_V^T \rightarrow \\ \leftarrow \vec{q}_V^T A \rightarrow \\ \leftarrow \vec{q}_V^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}_V^T A^{n-1} \rightarrow \end{bmatrix} \leftarrow \text{this is an } n \times n \text{ matrix}$$

← it will be full rank and invertible

5. Define $\vec{z} = T\vec{x} \Leftrightarrow \vec{x} = T^{-1}\vec{z}$ (this is called a basis transformation from $\vec{x}$ to $\vec{z}$, using T)

6. Using the definition of $\vec{z}$ in $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}u(t)$, you get, equivalently,

$$\frac{d\vec{z}}{dt} = \underbrace{TAT^{-1}}_{\hat{A}}\vec{z} + \underbrace{T\vec{b}}_{\hat{b}}u(t)$$

7. then:

→ $\hat{A} \triangleq TAT^{-1}$, with $\hat{b} \triangleq T\vec{b}$ will be in c.c.f!

→ Proof of the above procedure and claims:

→ from definition, $R_n^{-1} R_n = I$, or

$$\rightarrow \begin{matrix} R_n^{-1} \\ \left[\begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \leftarrow \vec{q}^T \rightarrow \end{array} \right] \end{matrix} \begin{matrix} R_n \\ \left[\begin{array}{c} | \quad | \quad | \quad \left(\right) \\ \vec{b} \quad A\vec{b} \quad A^2\vec{b} \quad \dots \quad A^{n-1}\vec{b} \\ | \quad | \quad | \quad | \end{array} \right] \end{matrix} = \begin{matrix} I \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & & \\ & \ddots & \ddots & \\ 0 & 0 & & 1 \end{array} \right] \end{matrix}$$

$$\rightarrow \text{Hence: } \vec{q}^T \begin{matrix} R_n \\ \left[\begin{array}{c} | \quad | \quad | \quad \left(\right) \\ \vec{b} \quad A\vec{b} \quad A^2\vec{b} \quad \dots \quad A^{n-1}\vec{b} \\ | \quad | \quad | \quad | \end{array} \right] \end{matrix} = [0, 0, \dots, 0, 1]$$

$$\Rightarrow \vec{q}^T \vec{b} = 0, \vec{q}^T A\vec{b} = 0, \dots, \vec{q}^T A^{n-2}\vec{b} = 0 \\ \text{and } \vec{q}^T A^{n-1}\vec{b} = 1$$

$$\rightarrow \text{also from definition, } T \triangleq \begin{matrix} \left[\begin{array}{c} \leftarrow \vec{q}^T \rightarrow \\ \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \end{array} \right] \Rightarrow \begin{bmatrix} \vec{q}^T \vec{b} \\ \vec{q}^T A\vec{b} \\ \vec{q}^T A^2\vec{b} \\ \vdots \\ \vec{q}^T A^{n-1}\vec{b} \end{bmatrix} = \underbrace{\vec{\hat{b}}}_{T\vec{b}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

→ Consider TA :

$$\rightarrow TA: \begin{matrix} \left[\begin{array}{c} \leftarrow \vec{q}^T \rightarrow \\ \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \end{array} \right] A = \begin{bmatrix} \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \leftarrow \vec{q}^T A^3 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \\ \leftarrow \vec{q}^T A^n \rightarrow \end{bmatrix}$$

→ notice that the top $(n-1)$ rows of $\hat{A}$ are just the last $(n-1)$ rows of T , i.e. the latter are shifted up by 1.

→ The last entry, $\vec{q}^T A^n$, can be expressed as a linear combination of the others, using the Cayley-Hamilton Theorem

→ C-H. Thm: A satisfies its own characteristic polynomial.

→ i.e., if $C(\lambda) = \lambda^n + a_n \lambda^{n-1} + \dots + a_1$ is the char. poly. of A .

then $A^n + a_n A^{n-1} + \dots + a_1 I = 0$, or

$$A^n = -a_n A^{n-1} - a_{n-1} A^{n-2} - \dots - a_1 I$$

$$\rightarrow \vec{q}^T A^n = -a_n \vec{q}^T A^{n-1} - a_{n-1} \vec{q}^T A^{n-2} - \dots - a_1 \vec{q}^T A$$

→ These two observations can be encapsulated in matrix form as:

$$\rightarrow TA = \begin{bmatrix} \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \leftarrow \dots \vec{q}^T A^3 \rightarrow \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \\ \leftarrow \vec{q}^T A^n \rightarrow \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \end{bmatrix} \begin{bmatrix} \leftarrow \vec{q}^T \rightarrow \\ \leftarrow \vec{q}^T A \rightarrow \\ \leftarrow \vec{q}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{q}^T A^{n-1} \rightarrow \end{bmatrix}$$

call this $\hat{A}$

→ thus we have $TA = \hat{A}T$ or $\hat{A} = TAT^{-1}$

we can write this form ONLY if T is invertible

→ and $(\hat{A}, \hat{b})$ are in c.c.f.

↳ this is proved on the next page.

→ A note about a fallacious proof:

→ in some places, a procedure is proposed that:

1. starts from some $(\hat{A}, \hat{b})$ in c.c.f. (with $a_1, \dots, a_n$ arbitrary)

2. builds $\hat{R}_n \triangleq [\hat{A}^{n-1}\hat{b}, \hat{A}^{n-2}\hat{b}, \dots, \hat{b}]$ and shows that it is lower triangular, with 1s on the diagonal,

3. defines $R_n \triangleq [\hat{A}^{n-1}\vec{b}, \hat{A}^{n-2}\vec{b}, \dots, \vec{b}]$

4. defines $T = \hat{R}_n R_n^{-1}$

and 5. claims that $\hat{A} = TAT^{-1}$

for example, choosing $a_1 = a_2 = \dots = a_n = 0$ would imply that $\hat{A}$, which is singular, is similar to any matrix A - which is obviously wrong.

because step 5 is NOT TRUE IN GENERAL for ARBITRARY CHOICES of $a_1, \dots, a_n$

→ This procedure DOES NOT PROVE that controllability of $(A, \vec{b}) \Rightarrow (TAT^{-1}, T\vec{b})$ is in c.c.f.

→ PROOF THAT T ON THE PREVIOUS PAGE IS INVERTIBLE

— To prove that T is full-rank (or non-singular), we will look at TR_n , and show that this is always non-singular.

→ Since A_n is non-singular (it is the controllability matrix), this implies that T must be non-singular, i.e., invertible.

— From the above, we know that:

— $T\vec{b} = \hat{\vec{b}}$, and

— $TA = \hat{A}T$,

— where $(\hat{A}, \hat{\vec{b}})$ are in CCF.

→ therefore:

$$\rightarrow TA\vec{b} = \hat{A}T\vec{b} = \hat{A}\hat{\vec{b}} = (\text{last col. of } \hat{A}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ a_n \end{bmatrix} \leftarrow (n-1)^{\text{th}} \text{ entry}$$

$$\rightarrow TA^2\vec{b} = TA(A\vec{b}) = \hat{A}TA\vec{b} = \hat{A} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_n \\ a_n^2 + a_{n-1} \end{bmatrix} \leftarrow (n-2)^{\text{th}} \text{ entry}$$

$$\rightarrow TA^3\vec{b} = TA(A^2\vec{b}) = \hat{A}TA^2\vec{b} = \hat{A} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_n \\ a_n^2 + a_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ a_n \\ a_n^2 + a_{n-1} \\ a_{n-2} + a_n a_{n-1} + a_n(a_n^2 + a_{n-1}) \end{bmatrix} \leftarrow (n-3)^{\text{th}} \text{ entry}$$

$\vdots$ all the way to

$$\rightarrow TA^{n-1}\vec{b} = \begin{bmatrix} 1 \\ * \\ \vdots \\ * \end{bmatrix} \leftarrow (1^{\text{st}} \text{ entry})$$

*s represent potential non-zero numbers.

— Using the above, we have:

$$TR_n = T \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{b}, A\vec{b}, \dots, A^{n-1}\vec{b} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ T\vec{b}, TA\vec{b}, TA^2\vec{b}, \dots, TA^{n-1}\vec{b} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{call this } L$$

L is a strictly lower triangular matrix, with 1 's on a diagonal $\Rightarrow$ it is full rank.

$\rightarrow$ therefore, we have $TR_n = L$, with R_n and L both full rank/invertible

$$\Rightarrow \boxed{\begin{array}{l} T \text{ itself has to be full rank and invertible, with} \\ T = R_n^{-1}L, \text{ and } T^{-1} = L^{-1}R_n \end{array}}$$

— Why are the above results useful?

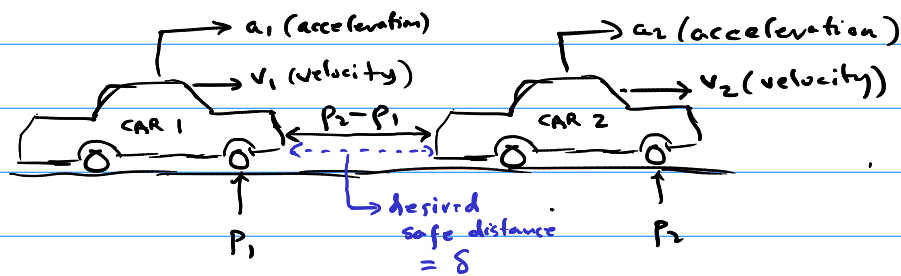
— Because they tell us that if a system is controllable, we can always devise feedback to place its eigenvalues wherever we want.

— but we don't necessarily have to put it in c.c.f first to do eigenvalue placement.

— we can do it directly — ie, writing out $C(s) \triangleq \det(A - \vec{b}\vec{k}^T - \lambda I)$ and matching its coefficients w those of $(\lambda - \lambda_1) \dots (\lambda - \lambda_n)$

↳ leads to a linear system of eqns in the unknowns $k_1, \dots, k_n$, which can be solved ^(eg.) numerically.

— Example: co-operative adaptive cruise control



CAR 1

$$v_1 = \frac{dp_1}{dt}$$

$$a_1 = \frac{dv_1}{dt}$$

CAR 2

$$v_2 = \frac{dp_2}{dt}$$

$$a_2 = \frac{dv_2}{dt}$$

$$\frac{d}{dt} (p_2 - p_1) = v_2 - v_1$$

call this $x_1 + \delta$, or $x_1 = p_2 - p_1 - \delta$

↳ will be 0 if cars at desired safe distance

$$\frac{d}{dt} (v_2 - v_1) = a_2 - a_1$$

↓

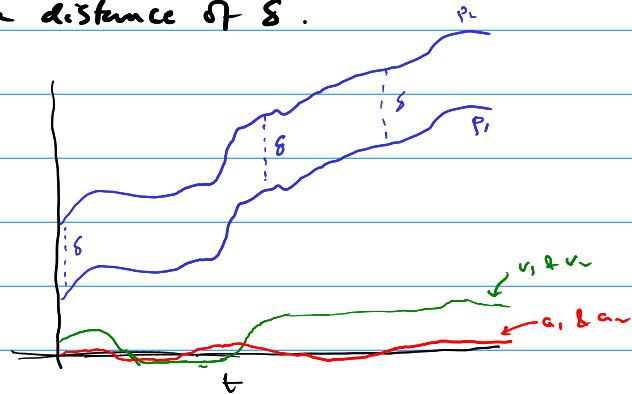
$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_A \vec{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} u(t)$$

- Suppose, at $t=0$: $x_1=0 \Rightarrow p_2 - p_1 = \delta$ ^{desired distance}
 $x_v=0 \Rightarrow v_2 = v_1$ ← same velocities

→ and car 1's driver (somehow) perfectly matches car 2's acceleration: i.e., $a_1(t) \equiv a_2(t) \Rightarrow u(t)=0$.

→ then $\frac{d\vec{x}}{dt} = 0$, and nothing changes.

→ they both go at the same velocities, keeping a distance of δ .



- Now, suppose there is a small acceleration error made by the driver of car 1: $a_1(t) = a_2(t) + \Delta a(t) \Rightarrow u(t) = \Delta a(t)$

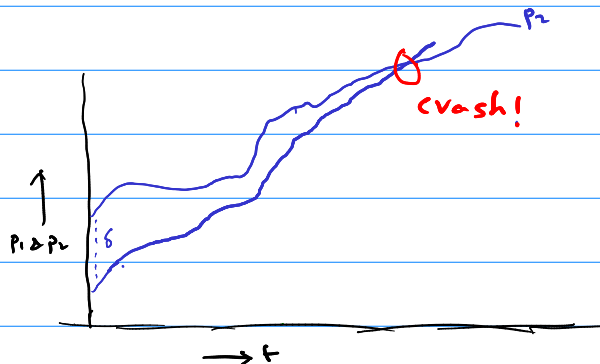
→ solving the system: we have

$$\rightarrow \Delta v(t) = v_2(t) - v_1(t) = \int_0^t \Delta a(z) dz$$

$$\rightarrow \Delta p(t) = p_2(t) - p_1(t) - \delta = \int_0^t \Delta v(z) dz$$

→ example: $\Delta a = \text{small constant } \epsilon$

$\Rightarrow \Delta v(t) = \epsilon t$, and $\Delta p(t) = \frac{\epsilon t^2}{2}$ → keeps increasing without bounds, cars will hit each other if $\epsilon < 0$



— How can feedback help?

$$\rightarrow A \mapsto \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} ; \quad c(\lambda) = \lambda(\lambda + k_2) + k_1 = 0 \Rightarrow \lambda^2 + k_2\lambda + k_1 = 0$$

$\downarrow$ position feedback $\downarrow$ velocity feedback
 Dappler radar provides both.

$$\lambda_{1,2} = -\frac{k_2}{2} \pm \frac{1}{2} \sqrt{k_2^2 - 4k_1}$$

$\rightarrow$ make $k_2 > 0, k_1 > 0$
 $\rightarrow \lambda_{1,2}$ both will have -ve real parts

$$\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_2 = \lambda x_1$$

$$-(k_1 x_1 + k_2 x_2) = \lambda x_2$$

$$\Rightarrow -k_1 x_1 - k_2 \lambda x_1 = \lambda^2 x_1 \Rightarrow (\lambda^2 + k_2 \lambda + k_1) x_1 = 0$$

$$\dot{\vec{x}} = A\vec{x} + \vec{b}u$$

$$P\dot{\vec{z}} = AP\vec{z} + \vec{b}u$$

$$\dot{\vec{z}} = \tilde{A}\vec{z} + \tilde{b}u$$

$$\tilde{A} = P\tilde{A}P^{-1}$$

$$\tilde{z} = P^{-1}\vec{x} \Leftrightarrow \vec{x} = P\tilde{z} \quad \checkmark$$

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} ; \quad P^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

say $x_1 = 1$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \int_0^t e^{\lambda_1(t-z)} \Delta a(z) dz \\ \lambda_2 \int_0^t e^{\lambda_2(t-z)} \Delta a(z) dz \end{bmatrix}$$

$$\text{if } \Delta a(t) \equiv \epsilon$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \frac{\epsilon}{\lambda_1} [1 - e^{\lambda_1 t}] \\ \epsilon [1 - e^{\lambda_2 t}] \end{bmatrix}$$

$\rightarrow$ always in the range $[0, \epsilon/\lambda_1]$

$$\int_0^t e^{\lambda(t-z)} dz = \frac{1}{\lambda} [1 - e^{\lambda t}]$$

$\rightarrow$ always in the range $[0, \epsilon]$

$$\text{hence } x_1(t) = \frac{\epsilon}{\lambda_1} [1 - e^{\lambda_1 t}] + \epsilon [1 - e^{\lambda_2 t}] \in [0, 1 + \frac{1}{\lambda_1}]$$

$\uparrow$
 make λ_1 large
 this $\rightarrow 1$

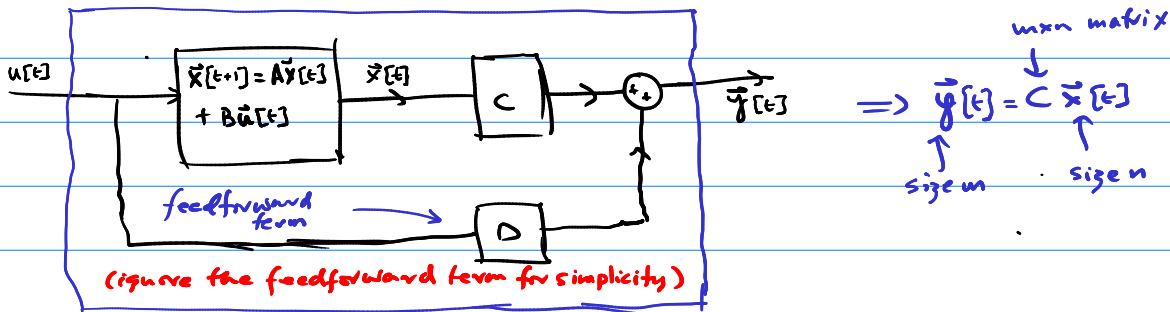
OBSERVABILITY AND OBSERVERS

→ So far, we have mostly been treating $\vec{x}(t)$ as the output $\vec{y}(t)$

→ but in most practical situations, the outputs are different from the state.

→ recall our output equation: $\vec{y}[k] = C \vec{x}[k] + D \vec{u}[k]$

(BACK TO DISCRETE SYSTEMS)



→ Q: from measurements of the output (and knowing the input)
can you infer the state? (especially if $m < n$)

→ If yes, the system (A, B, C) is called OBSERVABLE.

→ Say, $m=1$ (just one output), while $\vec{x} \in \mathbb{R}^n$, $n > 1$

$$y[k] = \begin{matrix} \vec{c}^T \\ \downarrow \\ [c_1 \ c_2 \ \dots \ c_n] \end{matrix} \vec{x}[k] \rightarrow 1 \times n$$

$$y[0] = \vec{c}^T \vec{x}[0]$$

$$y[1] = \vec{c}^T \vec{x}[1] = \vec{c}^T (A \vec{x}[0] + B \vec{u}[0])$$

$$y[2] = \vec{c}^T \vec{x}[2] = \vec{c}^T (A^2 \vec{x}[0] + AB \vec{u}[0] + B \vec{u}[1])$$

⋮

$$y[i] = \vec{c}^T \vec{x}[i] = \vec{c}^T \left(A^i \vec{x}[0] + \sum_{j=1}^i (A^{i-j} B \vec{u}[j-1]) \right)$$

$$y[n] = \vec{c}^T \vec{x}[n] = \vec{c}^T \left(A^n \vec{x}[0] + \sum_{j=1}^n (A^{n-j} B \vec{u}[j-1]) \right)$$

→ Now, say $u[t] \equiv 0$ — i.e., no input.

→ might you still be able to recover $\vec{x}[0], \vec{x}[1], \text{etc.}$ from $y[t]$?

→ boils down to recovery of just $\vec{x}[0]$, from which $\vec{x}[1], \vec{x}[2], \text{etc.}$ can be recovered

$$\begin{aligned} y[0] &= \vec{c}^T \vec{x}[0] \\ y[1] &= \vec{c}^T \vec{x}[1] = \vec{c}^T A \vec{x}[0] \\ y[2] &= \vec{c}^T \vec{x}[2] = \vec{c}^T A^2 \vec{x}[0] \\ &\vdots \\ y[i] &= \vec{c}^T \vec{x}[i] = \vec{c}^T A^i \vec{x}[0] \\ y[n] &= \vec{c}^T \vec{x}[n] = \vec{c}^T A^n \vec{x}[0] \end{aligned}$$

$$\rightarrow \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n] \end{bmatrix} = \begin{bmatrix} \leftarrow \vec{c}^T \rightarrow \\ \leftarrow \vec{c}^T A \rightarrow \\ \leftarrow \vec{c}^T A^2 \rightarrow \\ \vdots \\ \leftarrow \vec{c}^T A^{n-1} \rightarrow \end{bmatrix} \vec{x}[0]$$

↳ observability matrix O

→ if O is full rank (invertible), then yes, otherwise no.

$$\vec{x}[0] = O^{-1} \begin{bmatrix} y[0] \\ \vdots \\ y[n] \end{bmatrix}$$

→ more generally, if $m > 1$ (several outputs), $\vec{y} = C \vec{x}$

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \begin{matrix} \xleftarrow{n} \\ \xrightarrow{k} \end{matrix}$$

← must be full rank, i.e., $\text{rank} = n$

always

→ Observability boils down to finding just the IC, which we don't know

→ we know $A, B, C, u(t)$ and $y(t)$

↳ CLARIFY THIS POINT IN THE BEGINNING.

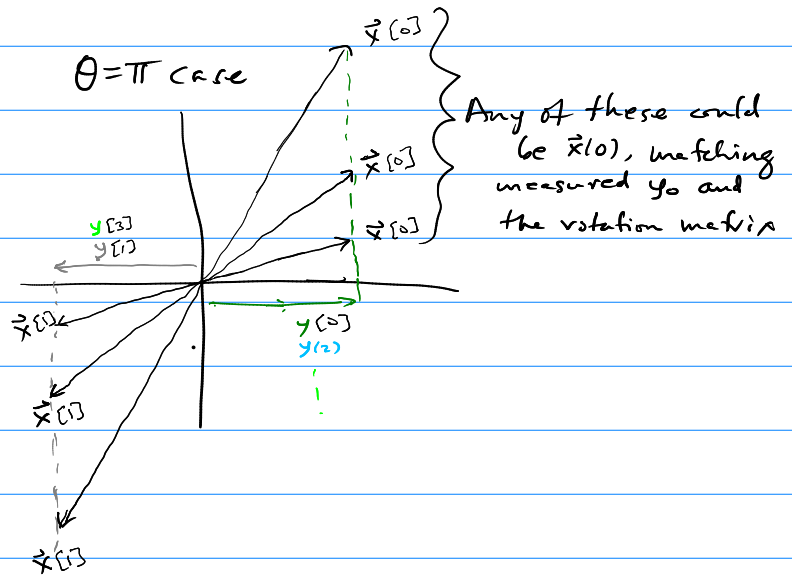
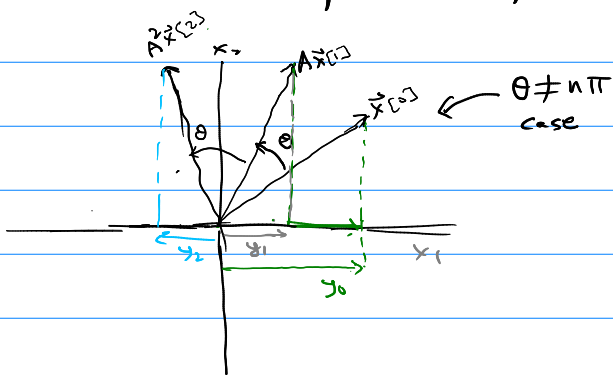
EXAMPLE:
$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}^A \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$$

→ this is a rotation matrix

$$y[t] = x_1[t] = \underbrace{[1 \ 0]}_{\underline{z}^T} \vec{x}[t]$$

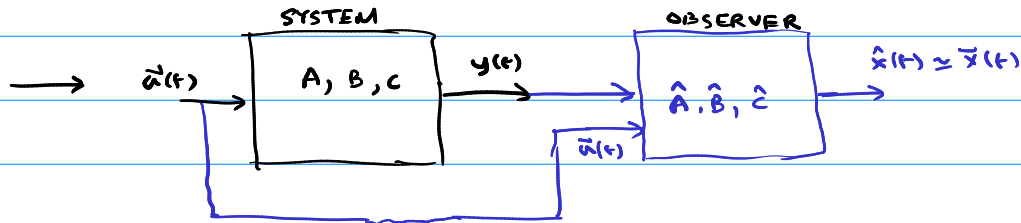
$$\begin{bmatrix} \leftarrow \underline{z}^T \rightarrow \\ \leftarrow \underline{z}^T A \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos(\theta) & -\sin(\theta) \end{bmatrix} : \begin{array}{l} \text{rank} = 2 \text{ if } \sin(\theta) \neq 0 \\ \text{rank} = 1 \text{ if } \theta = 0, \pi, 2\pi, \dots \end{array}$$

Graphical interpretation



OBSERVERS

→ A NEW SYSTEM (that you set up) TO ESTIMATE $\hat{x}(t)$ from the output



→ Form of observer: $\hat{x}[t+1] = A\hat{x}[t] + B\hat{u}[t] + \underbrace{L(C\hat{x}[t] - \hat{y}[t])}_{\substack{n \times m \\ m \times n}}$

if $\hat{x}[t] \approx \hat{y}[t]$, then
this term will be 0
and the observer
will be just the original
system.

— How good is the observer at doing its job (estimating $\hat{x}[t]$ well)?

— define the error $\tilde{e}(t) \triangleq \hat{x}[t] - \tilde{x}[t]$

— then $\tilde{e}[t+1] = \hat{x}[t+1] - \tilde{x}[t+1]$

$$= A(\hat{x}[t] - \tilde{x}[t]) + LC(\hat{x}[t] - \tilde{x}[t])$$

$$\Rightarrow \tilde{e}[t+1] = A\tilde{e}[t] + LC\tilde{e}[t] = (A+LC)\tilde{e}[t]$$

→ we would like $\tilde{e}[t]$ to go to 0

⇒ choose L to make the eigenvalues of $(A+LC)$ stable!

→ exactly like feedback for controllability!

→ recall: $(A+BK)$ for controllability.

→ you can re-use all that algebra and results: treat like $-B$
treat like K

— e.v.s of $(A+LC) =$ e.v.s of $(A+LC)^T = A^T + C^T L^T$

→ i.e., can always place eigenvalues if $(A^T, -C^T)$ controllable

$$\Rightarrow [A^T, A^T C^T, \dots, A^T C^T, C^T] \text{ has full rank}$$

$\Rightarrow [(A^T)^{k-1} C^T, (A^T)^{k-2} C^T, \dots, A^T C^T, C^T]^T$ has full rank

$$\hookrightarrow = \begin{bmatrix} - & CA^{k-1} & - \\ - & CA^{k-2} & - \\ & \vdots & \\ - & CA & - \\ - & C & - \end{bmatrix} \leftarrow \text{Simply the Observability criterion for the system}$$

Example: same rotation matrix example

$$- \quad \begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}^A \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$$

$$y[t] = x_1[t] = \underbrace{[1 \ 0]}_{\tilde{C}^T} \tilde{x}[t]$$

$$- \text{ try } \theta = \pi/2 \text{ as a special case: } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{C}^T = [1 \ 0]$$

$$\rightarrow A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (\tilde{C}^T)^T = \tilde{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\rightarrow \text{e.v.s of } A^T: \quad \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm j \leftarrow \text{magnitude} = 1, \text{ (discrete-time) BIBO unstable}$$

$$\rightarrow \text{feedback: } \tilde{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \Rightarrow \tilde{C} \tilde{l}^T = \begin{bmatrix} l_1 & l_2 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^T + \tilde{C} \tilde{l}^T = \begin{bmatrix} l_1 & 1+l_2 \\ -1 & 0 \end{bmatrix}$$

$$\text{eigenvalues: } (l_1 - \lambda)(-\lambda) + (1+l_2) = 0$$

$$\Rightarrow \lambda^2 - l_1 \lambda + (1+l_2) = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{l_1 \pm \sqrt{l_1^2 - 4(1+l_2)}}{2}$$

$$\Rightarrow \underline{\lambda_1 + \lambda_2 = l_1} \text{ and } \lambda_1 - \lambda_2 = \pm \sqrt{l_1^2 - 4(1+l_2)}$$

$$\Rightarrow (\lambda_1 - \lambda_2)^2 = l_1^2 - 4(1+l_2)$$

$$= (\lambda_1 - \lambda_2)^2 - (\lambda_1 + \lambda_2)^2 = -4(1+l_2)$$

$$\Rightarrow -4\lambda_1 \lambda_2 = -4(1+l_2)$$

$$\Rightarrow \underline{\lambda_2 = \lambda_1 \lambda_2 - 1}$$

→ choose any 2 stable λ s (respecting complex conjugacy)
and set λ_1 & λ_2

→ what if $\theta = \pi$ (unobservable case)?

$$\rightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; A^T = A$$

$$A^T + \bar{c} \bar{c}^T = \begin{bmatrix} -1 + \lambda_1 & \lambda_2 \\ 0 & -1 \end{bmatrix}$$

$$\det: -(\lambda+1)(-1+\lambda_1-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda_1 = -1}, \lambda_2 = \lambda_1 - 1$$

→ BIBO unstable and cannot be changed by choosing $\bar{c}$.

→ USING OBSERVERS FOR MORE ACCURATE POSITIONING & NAVIGATION

→ recall the accelerating car

discrete-time example from the

lecture on controllability:

Terminology: $x[k] \equiv x(kT)$; $v[k] \equiv v(kT)$; $a[k] \equiv a(kT)$

→ eigenvalues: $(1 - \lambda^2) = 0 \Rightarrow \lambda_{1,2} = \pm 1 \rightarrow$ **BIBO UNSTABLE**

→ TYPICAL FOR CARS, PLANES, ETC.

→ CONSIDER the scenario:

→ the car is NOT feedback stabilized

— there is (inevitable) error in the accel: $a[k] = a_d[k] + \Delta a[k]$

→ we estimate its state using an observer:

$$\rightarrow \hat{x}[k+1] = A \hat{x}[k] + \bar{b} a_d[k] \rightarrow \text{same as system, no } \bar{c} \bar{c}^T \text{ term}$$

→ we don't know $\Delta a[k]$

→ the estimation error follows:

$$\bar{e}[k+1] = A \bar{e}[k] + \bar{b} \Delta a[k] \leftarrow \text{unstable} \Rightarrow \bar{e}(t)$$

grows larger and larger even if $\Delta a(k)$ remains small.

→ but ^{if} we are able to measure the position of the actual car, say using GPS measurements:

$$y[t] = \underbrace{[1 \ 0]}_{\vec{c}^T} \hat{x}[t] + \Delta p[t]$$

→ GPS measurement error, also not precisely known, but small.

→ Our observer becomes:

$$\vec{l} (\vec{c}^T \hat{x} - y[t])$$

$$\begin{aligned} \hat{x}[t+1] &= A \hat{x}[t] + \vec{b} a_d[t] + \vec{l} (\vec{c}^T \hat{x} - y[t]) \\ &= A \hat{x}[t] + \vec{b} a_d[t] + \vec{l} \vec{c}^T \hat{x}[t] - \vec{l} \vec{c}^T \vec{x}[t] - \vec{l} \Delta p[t] \\ &= (A + \vec{l} \vec{c}^T) \hat{x}[t] + \vec{l} \vec{c}^T (\hat{x}[t] - \vec{x}[t]) + \vec{b} a_d[t] - \vec{l} \Delta p[t] \end{aligned}$$

Hence the error $\vec{e}[t] \triangleq \hat{x}[t] - \vec{x}[t]$ obeys

$$\vec{e}[t+1] = A \vec{e}[t] + \vec{l} \vec{c}^T \vec{e}[t] + \underbrace{(\vec{b} a_d[t] - \vec{l} \Delta p[t])}_{\text{due to errors, but small}}$$

$$\Rightarrow \vec{e}[t+1] = (A + \vec{l} \vec{c}^T) \vec{e}[t] + \underbrace{(\vec{b} a_d[t] - \vec{l} \Delta p[t])}_{\text{always small}}$$

→ Now, if $(A, \vec{c})$ are observable, you can _{always} choose $\vec{l}$ to stabilize the observer

→ which means that the estimation error can always be kept bounded and small, even taking into account small errors in GPS & acceleration

↳ this is what all navigational systems today do!

→ Actually, they use an additional improvement, whereby $\vec{l}$ changes w/ t (i.e., becomes $\vec{l}[t]$)

→ the updates are used to minimize the error due to GPS/pedal errors, then noise, etc, even further

↳ this is the famous **KALMAN FILTER**, the gold standard for accurate navigational estimation

— WHAT IT ACHIEVES:

- open-loop position/velocity estimation $\rightarrow$ blows up errors $\therefore$ unstable
- estimation simply by GPS also has errors

— using a well-designed observer, you can make much better estimates than either of these 2 alone. $\leftarrow$ quite remarkable

— and then, you can use this good estimate^{of} $\hat{x}$ to feed back to the car (via controllability) to make its response stable.

$\rightarrow$ THIS IS WHAT IS GOING ON
IN AUTONOMOUS VEHICLES
(TESLA PICTURE)

— PICTURES AND HISTORY OF KALMAN.

$\rightarrow$ ALSO: FEEDBACK STABILIZED LEVITRON DEMOS (earlier)