$$J_u = \left.\frac{\partial \vec{f}}{\partial \vec{u}}\right|_{\vec{x}^*, \vec{u}^*} = \begin{bmatrix} \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial b} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m\ell} \end{bmatrix}$$

2x1 vector

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} v_\theta \\ -\frac{g}{\ell}\sin(\theta) - \frac{k}{m}v_\theta + \frac{b}{m\ell} \end{bmatrix}$$

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\theta, v_\theta; b) \\ f_2(\theta, v_\theta, b) \end{bmatrix}$$

$$f_1(\cdots) \equiv v_\theta$$

$$f_2(\cdots) = -\frac{g}{\ell}\sin(\theta) - \frac{k}{m}v_\theta + \frac{b}{m\ell}$$

$$J_x \triangleq \left.\frac{\partial \vec{f}}{\partial \vec{x}}\right|_{x^*, u^*} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta}, & \frac{\partial f_1}{\partial v_\theta} \\ \frac{\partial f_2}{\partial \theta}, & \frac{\partial f_2}{\partial v_\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(\pi) & -\frac{k}{m} \end{bmatrix}$$

$$+\frac{g}{\ell}$$

$$\vec{x}^* = \begin{bmatrix} \theta_2^* \\ v_\theta^* \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \theta \\ v_\theta \end{bmatrix}$$

EIGENDECOMPOSITION : $A = P \Lambda P^{-1} \iff AP = P\Lambda$

invertible

$$A = P \Lambda P^{-1} \iff$$

eigenvalues

$$\begin{array}{ccc} P & & \\ \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} & \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} & \begin{bmatrix} & & \\ & P^{-1} & \\ & & \end{bmatrix} \end{array}$$

eigenvectors

$$AP = P\Lambda$$

$$\begin{array}{cccc} & P & & AP \\ A & \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & p_2 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} & = & \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ A\vec{p}_1 & A\vec{p}_2 & \cdots & A\vec{p}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \end{array}$$

$$\begin{array}{cccc} P & \Lambda & & \\ \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{p}_1 & p_2 & \cdots & \vec{p}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} & \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} & = & \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1\vec{p}_1 & \lambda_2\vec{p}_2 & \cdots & \lambda_n\vec{p}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \end{array}$$

$$\boxed{A\vec{p}_i = \lambda_i \vec{p}_i}$$

$$A\vec{p} = \lambda \vec{p}$$

DIAGONALIZING $\frac{d}{dt}\delta\vec{x}(t) = A\,\Delta\vec{x}(t) + B\,\vec{u}(t)$ using $A = P\Lambda P^{-1}$

$$\frac{d}{dt}\delta\vec{x}(t) \overset{\text{s.s.r}}{=} \underbrace{A}_{P\Lambda P^{-1}}\,\Delta\vec{x}(t) + B\,\vec{u}(t) \;\Rightarrow\; P^{-1}\frac{d\Delta\vec{x}(t)}{dt} = P^{-1}\left(P\Lambda P^{-1}\Delta\vec{x}(t) + B\vec{u}(t)\right)$$

$$\frac{d}{dt}\Big[\underbrace{P^{-1}\Delta\vec{x}(t)}_{\Delta\vec{y}(t)}\Big] = \Lambda\underbrace{P^{-1}\Delta\vec{x}(t)}_{\Delta\vec{y}(t)} + P^{-1}B\,\vec{u}(t)$$

$$\Delta\vec{y}(t) \triangleq P^{-1}\Delta\vec{x}(t) \iff \Delta\vec{x}(t) = P\Delta\vec{y}(t)$$

$$\boxed{\frac{d}{dt}\Delta\vec{y}(t) = \Lambda\,\Delta\vec{y}(t) + \underbrace{(P^{-1}B)\,\vec{u}(t)}_{\vec{b}(t)}}$$

$$\frac{d}{dt}\begin{bmatrix}\Delta y_1(t)\\ \Delta y_2(t)\\ \vdots\\ \Delta y_n(t)\end{bmatrix} = \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}\begin{bmatrix}\Delta y_1(t)\\ \Delta y_2(t)\\ \vdots\\ \Delta y_n(t)\end{bmatrix} + \begin{bmatrix}b_1(t)\\ \vdots\\ \vdots\\ b_n(t)\end{bmatrix}$$

$$\frac{d}{dt}\Delta y_1(t) = \lambda_1\Delta y_1 + b_1(t)$$

$$\frac{d}{dt}\Delta y_2(t) = \lambda_2\Delta y_2 + b_2(t)$$

$$\vdots$$

$$\frac{d}{dt}\Delta y_n(t) = \lambda_n\Delta y_n + b_n(t)$$

EIGENVALUES OF LINEARIZED PENDULUM

$$A = \begin{bmatrix} 0 & 1 \\ -g/l & -\frac{k}{m} \end{bmatrix}$$

$$\Delta\vec{x}[t+1] = A\Delta\vec{x}[t] + B\Delta\vec{u}[t]$$

$t=0$

$t=1$

$t=2$

$\vdots$

t