DERIVATION OF THE SOLUTION OF $\dot{x} = ax + b\,u(t)$

$$\frac{dx}{dt} = a x(t) + b\,u(t)$$

$F(t)$ = integrating factor $= e^{-\int a\,dt}$

$= e^{-at}$

$$F(t)\frac{dx}{dt} = a F(t) x(t) + b F(t) u(t)$$

$$F(t)\frac{dx}{dt} - a F(t) x(t) = b F(t) u(t)$$

$$\frac{d}{dt}\left[F(t) x(t)\right] = \left[\frac{d}{dt} F(t)\right] x(t) + F(t)\frac{d}{dt} x(t)$$

$-a e^{-at}$

$$= F(t)\frac{d}{dt} x(t) - a F(t) x(t)$$

$$\frac{d}{dt}\left[e^{-at} x(t)\right] = b e^{-at} u(t)$$

$$e^{-at} x(t) = \int e^{-at} b\,u(t)\,dt$$

limits: 0 to s: $\left[e^{-at} x(t)\right]_0^s$

$$= \int_0^s e^{-at} b\,u(t)\,dt$$

limits: 0 to s: $\left[e^{-at}x(t)\right]_0^s = \int_0^s e^{-at} b\, u(t)\, dt$

$$e^{-as}x(s) - x(0) = \int_0^s e^{-at} b\, u(t)\, dt \Rightarrow e^{-as}x(s) = x(0) + \int_0^s e^{-at} b\, u(t)\, dt$$

$$\boxed{x(s) = e^{+as}\left[x(0) + \int_0^s e^{-at} b\, u(t)\, dt\right] = \boxed{x(0)\,e^{as} + \int_0^s e^{a(s-t)}\, b\, u(t)\, dt}}$$

change $t \to \tau$, $s \to t$

$$x(t) = x(0)e^{at} + \int_0^t e^{a(t-\tau)}\, b\, u(\tau)\, d\tau$$

$$x[t+1] = a\,x[t] + b\,u[t] \qquad \text{IC: } x[0] \text{ given}$$

$t=0$: $\quad x[1] = a\,x[0] + b\,u[0]$

$t=1$: $\quad x[2] = a\,x[1] + b\,u[1] = a^2 x[0] + ab\,u[0] + b\,u[1]$

$t=2$: $\quad x[3] = a\,x[2] + b\,u[2] = a^3 x[0] + a^2 b\,u[0] + ab\,u[1] + b\,u[2]$

$\vdots$

t $\quad x[t] = a^t x[0] + a^{t-1} b\,u[0] + a^{t-2} b\,u[1] + \cdots + b\,u[t-1]$

$$\Rightarrow x[t] = a^t x[0] + \sum_{i=1}^{t} a^{t-i} b\,u[i-1]$$