THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

1. INTRODUCTION AND PRELIMINARIES
2. ROOTS OF 1 ON THE COMPLEX PLANE
3. THE DFT MATRIX
4. PROPERTIES, INCLUDING ITS INVERSE
5. DFTs OF REAL SEQUENCES/DATA, INCLUDING COMPLEX CONJUGACY PROPERTIES
6. FREQUENCY INTERPRETATION OF DFT SIGNALS, D.T. BAND LIMITING, ...
7. APPLICATION TO LTI SYSTEMS: DFTs OF IMPULSE RESPONSES.

---

1. DFTs: INTRODUCTION

- → THE DFT IS JUST A MATRIX (SQUARE MATRIX)
  - → BUT A VERY SPECIAL MATRIX
    - → FILLED WITH (SPECIAL) COMPLEX NUMBERS
    - → INVERTIBLE (AND ITS INVERSE IS "ALMOST ITSELF")
- → WIDELY USEFUL
  - → CONVERT "TIME-DOMAIN" TO "FREQUENCY DOMAIN" AND VICE-VERSA
    - → DIGITAL SIGNAL PROCESSING
      - → FIND FREQUENCY SPECTRUM
      - → PERFORM FILTERING
      - → FOURIER ANALYSIS / SPECTROGRAMS / ETC.
    - → IMAGE COMPRESSION
      - → JPEG IMAGES (DCT)
    - → FAST COMPUTATION: FFT
    - → SHOW LIVE DEMO + SLIDES
  - → CAN BE USED FOR A SPECIAL KIND OF INTERPOLATION (BAND-LIMITED INTERPOLATION)
    - → SIMILAR TO SINC() INTERPOLATION
  - → CONNECTIONS WITH GLOBAL POLYNOMIAL INTERPOLATION AND VANDERMONDE MATRIX

→ PRELIMINARIES:

→ COMPLEX CONJUGATE NOTATION: BAR OVER NUMBER/VECTOR/MATRIX: $\bar{a}$, $\bar{x}$, $\bar{A}$

→ HERMITIAN: MEANS COMPLEX CONJUGATE TRANSPOSE

→ DENOTED BY * or ^{H} SUPERSCRIPT

→ $1^* = 1$; $(1+j)^* = (1-j)$

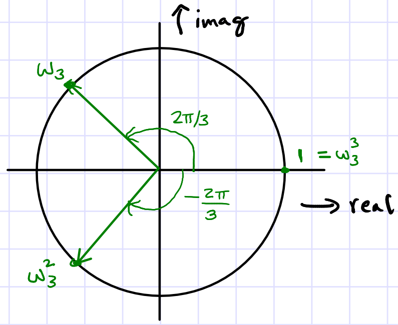
→ $\begin{bmatrix} 1 & e^{j\pi} \\ 2 & 1+j \end{bmatrix}^* = \begin{bmatrix} 1 & 2 \\ e^{-j\pi} & 1-j \end{bmatrix}$

---

2. ROOTS OF 1 ON THE COMPLEX PLANE

→ $\sqrt{1} = \pm 1$, FOCUS ON THE LESS OBVIOUS ONE: -1.

→ $\sqrt[3]{1} = 1,\ e^{+j\frac{2\pi}{3}},\ e^{-j2\pi/3}$



$\left(e^{j2\pi/3}\right)^2 = e^{j\frac{2\pi}{3}\cdot 2}$

$\left(e^{j\frac{2\pi}{3}}\right)^3 = e^{j2\pi} = 1.$

— FOCUS ON THE ONE w/ +ve angle:

→ $\omega_3 \triangleq e^{j2\pi/3}$

→ $\omega_3^3 = 1$

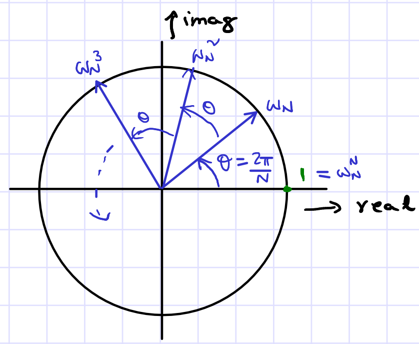
→ $\sqrt[N]{1} =$

small

"omega N"

→ $\omega_N \triangleq e^{j\frac{2\pi}{N}}$

→ $\omega_N^N = 1$



---

3. THE DFT MATRIX

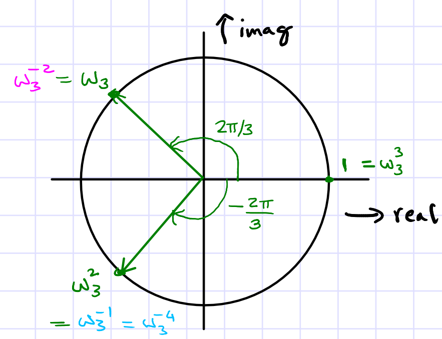
→ CONVENTION: ROW/COL INDICES i, j GO FROM $0, \ldots, N-1$

→ DFT MATRIX F_N: $(i,j)^{th}$ entry $= \omega_N^{-ij}$

→ EXAMPLE: $N=3$

$$F_3 = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \omega_3^{-0\times 0} & \omega_3^{-0\times 1} & \omega_3^{-0\times 2} \\ \omega_3^{-1\times 0} & \omega_3^{-1\times 1} & \omega_3^{-1\times 2} \\ \omega_3^{-2\times 0} & \omega_3^{-2\times 1} & \omega_3^{-2\times 2} \end{bmatrix} \end{array} = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3^{-1} & \omega_3^{-2} \\ 1 & \omega_3^{-2} & \omega_3^{-4} \end{bmatrix} \end{array} \quad N=3$$

$\longleftarrow N=3 \longrightarrow$



→ DFT - VECTOR PRODUCT

→ say $\vec{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \triangleq F_N \vec{x}$

→ Then: $y_i = \sum_{j=0}^{N-1} w_N^{-ij} x_j$, $i = 0, \ldots, N-1$ (WILL BE USEFUL LATER)

---

4. PROPERTIES OF THE DFT

A. SYMMETRIC: $F_N^T = F_N$ $\left((i,j)^{th} \text{ entry is } w_N^{-ij} \right)$

B. FULL RANK ≡ INVERTIBLE

→ $F_N^{-1} = \frac{1}{N} F_N^*$ ← HERMITIAN ← INVERSE DFT ≡ IDFT

→ OR: $F_N F_N^* = N I_N$ (ALMOST UNITARY)

→ PROOF:

→ LET $A = F_N F_N^*$ ← WILL SHOW THIS IS $N I_N$

→ WHAT IS a_{ij}, the $(i,j)^{th}$ entry of A?

→ $a_{ij} = (i^{th} \text{ row of } F_N) \times (j^{th} \text{ col of } F_N^*)$

$$= \sum_{k=0}^{N-1} w_N^{-ik} w_N^{+jk} = \sum_{k=0}^{N-1} w_N^{k(j-i)} = \underbrace{\left[w_N^{(j-i)}\right]^0}_{\text{call this } p_{ij}} + \left[w_N^{(j-i)}\right]^1 + \cdots + \left[w_N^{(j-i)}\right]^{N-1}$$

$$= p_{ij}^0 + p_{ij}^1 + p_{ij}^2 + \cdots + p_{ij}^{N-1}$$

→ IF $i = j$, then $p_{ij} = w_N^0 = 1$

$\Rightarrow a_{ii} = N$

→ IF $i \neq j$:

→ $a_{ij} = p_{ij}^0 + p_{ij}^1 + p_{ij}^2 + \cdots + p_{ij}^{N-1}$

→ $p_{ij} a_{ij} = p_{ij}^1 + p_{ij}^2 + \cdots + p_{ij}^{N-1} + p_{ij}^N$

SUBTRACT

$\Rightarrow a_{ij} - p_{ij} a_{ij} = p_{ij}^0 - p_{ij}^N \Rightarrow a_{ij}(1 - p_{ij}) = 1 - p_{ij}^N \Rightarrow a_{ij} = \frac{1 - p_{ij}^N}{1 - p_{ij}}$

→ $p_{ij}^N = w_N^{N(i-j)} = 1$

$\Rightarrow a_{ij} = \frac{1 - p_{ij}^N}{1 - p_{ij}} = 0$

→ SUMMARY: $a_{ij} = \begin{cases} N, & i = j \\ 0, & i \neq j \end{cases} \Rightarrow A = F_N F_N^* = N I_N$

→ Q: GIVEN THAT $N I_n = F_N F_N^*$, WHAT PROPERTY DO THE ROWS/COLS OF F_N HAVE?

→ A: THEY ARE ORTHOGONAL (BUT NOT ORTHONORMAL)

→ ALSO: THE SUM OF EVERY ROW/COL (EXCEPT THE FIRST) $= 0$

→ SUM OF FIRST ROW/COL $= N$.

→ PROVE THIS YOURSELVES

C. $\text{IDFT} = \text{DFT}^{-1} = F_N^{-1} = \frac{1}{N} F_N^*$

→ IDFT – VECTOR PRODUCT

→ Say $\vec{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \triangleq \text{IDFT} * \vec{x} = \frac{1}{N} F_N^* \vec{x}$

→ Then: $y_i = \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{ij} x_j$, $i = 0, \dots, N-1$ (WILL BE USEFUL LATER)

---

5. DFTs OF REAL VECTORS: COMPLEX CONJUGACY PROPERTIES