THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

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1. DFTs : INTRODUCTION

- → THE DFT IS JUST A MATRIX (SQUARE MATRIX)
  - → BUT A VERY SPECIAL MATRIX
    - → FILLED WITH (SPECIAL) COMPLEX NUMBERS
    - → INVERTIBLE (AND ITS INVERSE IS "ALMOST ITSELF")
- → WIDELY USEFUL
  - → CONVERT "TIME-DOMAIN" TO "FREQUENCY DOMAIN" AND VICE-VERSA
    - → DIGITAL SIGNAL PROCESSING
      - → FIND FREQUENCY SPECTRUM
      - → PERFORM FILTERING
      - → FOURIER ANALYSIS / SPECTROGRAMS / ETC.
    - → IMAGE COMPRESSION
      - → JPEG IMAGES (DCT)
    - → FAST COMPUTATION : FFT
    - → SHOW LIVE DEMO + SLIDES
  - → CAN BE USED FOR A SPECIAL KIND OF INTERPOLATION (BAND-LIMITED INTERPOLATION)
    - → SIMILAR TO SINC( ) INTERPOLATION
  - → CONNECTIONS WITH GLOBAL POLYNOMIAL INTERPOLATION AND VANDERMONDE MATRIX

→ PRELIMINARIES:

→ COMPLEX CONJUGATE NOTATION: BAR OVER NUMBER/VECTOR/MATRIX: $\bar{a}$, $\bar{x}$, $\bar{A}$

→ HERMITIAN: MEANS COMPLEX CONJUGATE TRANSPOSE

→ DENOTED BY * or ^{H} SUPERSCRIPT

→ $1^* = 1$; $(1+j)^* = (1-j)$

→ $$\begin{bmatrix} 1 & e^{j\pi} \\ 2 & 1+j \end{bmatrix}^* = \begin{bmatrix} 1 & 2 \\ e^{-j\pi} & 1-j \end{bmatrix}$$

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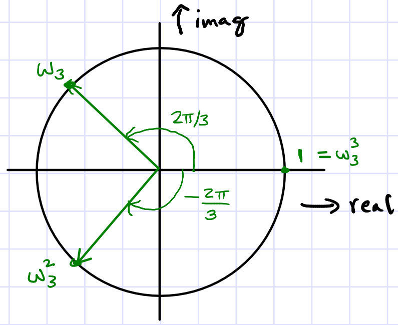
2. ROOTS OF 1 ON THE COMPLEX PLANE

→ $\sqrt{1} = \pm 1$, FOCUS ON THE LESS OBVIOUS ONE: -1.

→ $\sqrt[3]{1} = 1,\ e^{+j\frac{2\pi}{3}},\ e^{-j2\pi/3}$



$$\left(e^{j2\pi/3}\right)^2 = e^{j\frac{2\pi}{3}\cdot 2}$$

$$\left(e^{j\frac{2\pi}{3}}\right)^3 = e^{j2\pi} = 1.$$

— FOCUS ON THE ONE w/ +ve angle:

→ $\omega_3 \triangleq e^{j2\pi/3}$

→ $\omega_3^3 = 1$

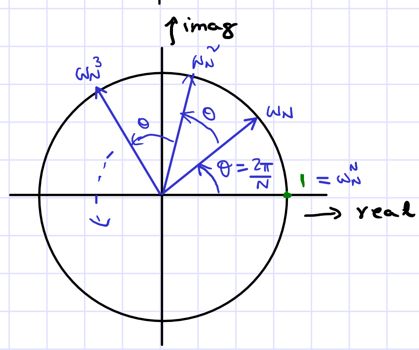
→ $\sqrt[N]{1} =$

small

"omega N"

→ $\omega_N \triangleq e^{j\frac{2\pi}{N}}$

→ $\omega_N^N = 1$



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3. THE DFT MATRIX

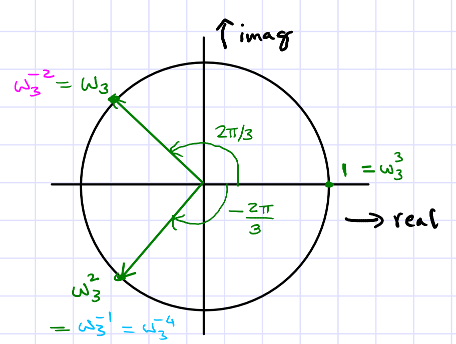
→ CONVENTION: ROW/COL INDICES i, j GO FROM $0, \ldots, N-1$

→ DFT MATRIX F_N: $(i,j)^{th}$ entry $= \omega_N^{-ij}$

→ EXAMPLE: $N=3$

$$F_3 = \begin{array}{c|c} i \downarrow\ j\rightarrow & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \omega_3^{-0\times 0} & \omega_3^{-0\times 1} & \omega_3^{-0\times 2} \\ \omega_3^{-1\times 0} & \omega_3^{-1\times 1} & \omega_3^{-1\times 2} \\ \omega_3^{-2\times 0} & \omega_3^{-2\times 1} & \omega_3^{-2\times 2} \end{bmatrix} \end{array} = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3^{-1} & \omega_3^{-2} \\ 1 & \omega_3^{-2} & \omega_3^{-4} \end{bmatrix} \end{array} \; N=3$$

$\longleftarrow N=3 \longrightarrow$



→ **DFT – VECTOR PRODUCT**

→ say $\vec{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \triangleq F_N \vec{x}$

→ Then: $y_i = \sum_{j=0}^{N-1} W_N^{-ij} x_j$, $i = 0, \ldots, N-1$ (WILL BE USEFUL LATER)

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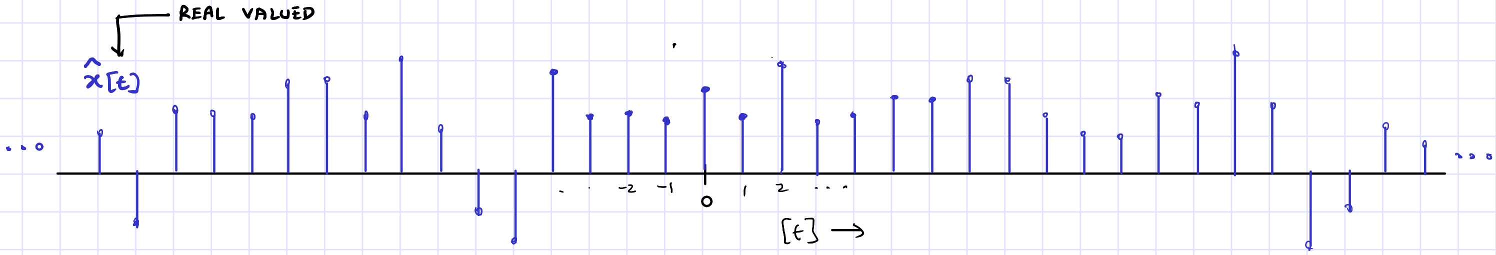
4. VECTORS FROM SEQUENCES

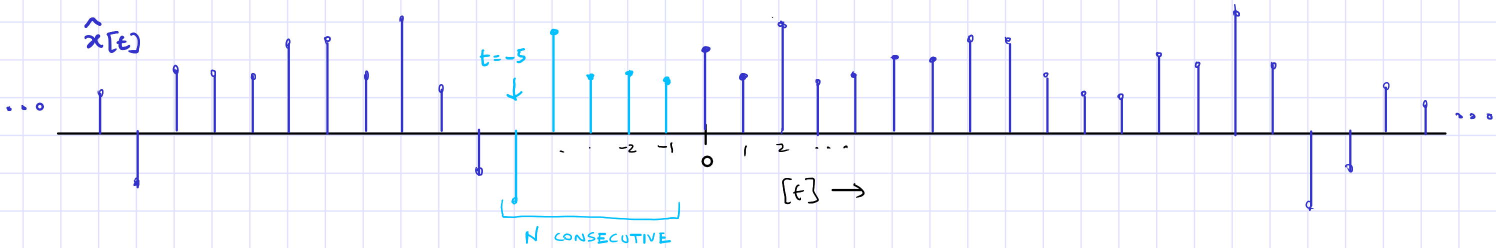
→ TURNING A CHUNK OF A SEQUENCE INTO A VECTOR

→ GIVEN SOME DISCRETE-TIME SEQUENCE : $\hat{x}[t]$, $t = -\infty, \cdots, -1, 0, 1, 2, \cdots \infty$



→ PICK ANY N CONSECUTIVE VALUES

→ EXAMPLE : $N=5$, $t = -5, -4, \cdots, -2, -1$.



→ MAKE A SIZE-N VECTOR OF THEM: $\vec{x} = \begin{bmatrix} x[-5] \\ x[-4] \\ x[-3] \\ x[-2] \\ x[-1] \end{bmatrix} \begin{matrix} \leftarrow x_0 \\ \leftarrow x_1 \\ \vdots \\ \\ \leftarrow x_{N-1} \end{matrix}$

→ WE CAN NOW RUN DFTs (OR OTHER VECTOR OPERATIONS) ON $\vec{x}$

→ THIS IS OFTEN USEFUL

→ E.G., THE SPECTROGRAM SHOWN EARLIER (DEMO/SLIDES) DOES JUST THIS, USING CONSECUTIVE CHUNKS OF THE INPUT EVERY TIME THE DISPLAY IS UPDATED.

→ UNLESS OTHERWISE SPECIFIED, WE WILL ASSUME BELOW THAT VECTORS CONTAIN CHUNKS FROM SEQUENCES, STARTING FROM $t=0$

→ e.g.:

$$\vec{u} = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix} \begin{matrix} \leftarrow u_0 \\ \leftarrow u_1 \\ \vdots \\ \leftarrow u_{N-1} \end{matrix}$$

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5. FREQUENCY INTERPRETATION OF THE DFT

→ OUR GOAL HERE IS TO TRY TO DEVELOP A FEELING/INTUITION FOR WHAT THE ENTRIES OF $\vec{X}$ REPRESENT

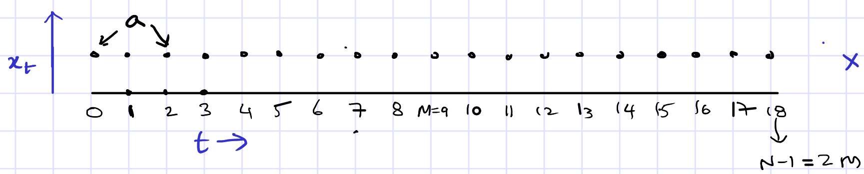
→ WE WILL ASSUME $\vec{x}$ (and $x[t]$) are always real.

→ THE FLOW IS LIKE THIS:

→ WE WILL MAKE UP SOME SIMPLE EXAMPLES (DC, SINUSOIDAL) of $\vec{x}$ or $x[t]$

→ WE WILL RUN THE DFT ON $\vec{x}$ and SEE WHAT $\vec{X}$ LOOKS LIKE

→ EXPERIMENT 1: $\vec{x}$ (or $x[t]$) is <u>DC</u>



$x[t]$ constant with $t \in \{0, \dots, N-1\}$

$\Rightarrow$ DC

$$\rightarrow \vec{X} = F_N \vec{x} \Rightarrow X_f = \sum_{t=0}^{N-1} W_N^{-tf} x_t$$

$$\rightarrow \underset{\uparrow \atop f=0}{X_0} = \sum_{t=0}^{N-1} x_t = \underline{Na}$$

$$\rightarrow \underset{\uparrow \atop f=1}{X_1} = a \underbrace{\sum_{t=0}^{N-1} W_N^{-t}}_{\text{CALL THIS } S}$$

$$\rightarrow \left.\begin{aligned} S &= 1 + W_N^{-1} + W_N^{-2} + \cdots + W_N^{-(N-1)} \\ W_N^{-1} S &= \quad\;\; W_N^{-1} + W_N^{-2} + \cdots + W_N^{-(N-1)} + W_N^{-N} \end{aligned}\right] \rightarrow \text{SUBTRACT}$$

$$\rightarrow S - W_N^{-1} S = 1 - W_N^{-N}$$

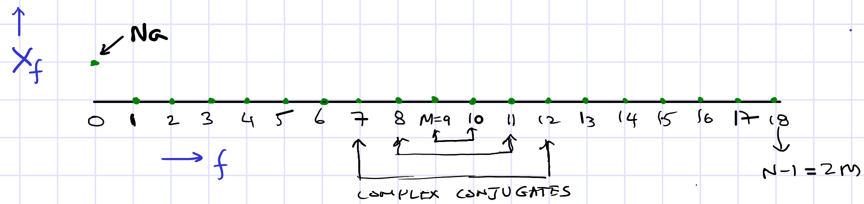
$$\Rightarrow (1 - W_N^{-1}) S = 1 - W_N^{-N} \Rightarrow S = \frac{1 - \overset{1}{W_N^{-N}}}{1 - W_N^{-1}} = 0$$

$$\Rightarrow \underline{X_1 = 0}$$

→ TRY $X_2, X_3, \dots, X_{N-1}$ ← ALL WILL BE 0

WORK IT OUT YOURSELF LATER.

→ SO OUR $\vec{X}$ LOOKS LIKE THIS :



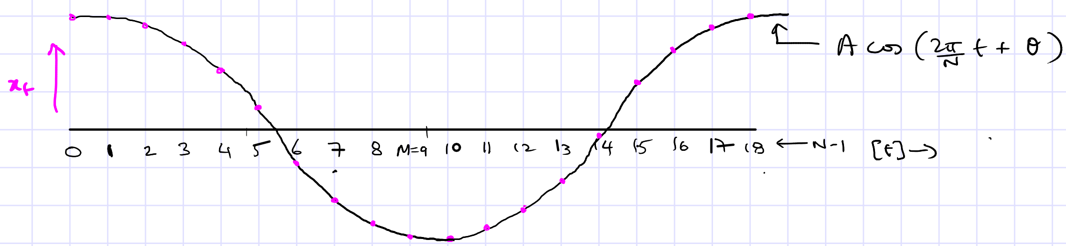
→ CONCLUSION : IF $\vec{x}\,/\,x[t]$ IS DC, $X_i = 0$ except X_0, which is N times the DC value.

→ NEXT EXPERIMENT : $\vec{x}\,/\,x[t]$ are samples of a sinusoid of period N

→ Say $x(t) = A\cos\left(\frac{2\pi}{N}t + \theta\right)$, $t \in \mathbb{R}$

N ← period

→ $x[t]$ = discrete sample = $A\cos\left(\frac{2\pi t}{N} + \theta\right)$, $t = 0, 1, 2, 3, \dots, N-1$



→ write $x[t]/x_t$ in COMPLEX / PHASOR NOTATION → $\frac{A}{2}e^{j\theta}$ IS THE PHASOR

$$\to x[t] = \frac{A}{2}\left(e^{j\left(\frac{2\pi}{N}t+\theta\right)} + e^{-j\left(\frac{2\pi}{N}t+\theta\right)}\right) = \frac{A}{2}\left(W_N^{t} e^{j\theta} + W_N^{-t} W_N^{-j\theta}\right)$$

$$\to \text{DFT of } \vec{x} : \quad \vec{X} = F_N \vec{x} \iff X_f = \sum_{t=0}^{N-1} W_N^{-tf} x_t$$

$t - tf = t(1-f)$

$$\Rightarrow X_f = \frac{A}{2}\sum_{t=0}^{N-1} W_N^{-tf}\left(W_N^{t} e^{j\theta} + W_N^{-t} W_N^{-j\theta}\right)$$

$$= \frac{A}{2}\left[\sum_{t=0}^{N-1} W_N^{t(1-f)} e^{j\theta} + \sum_{t=0}^{N-1} W_N^{-t(f+1)} W_N^{-j\theta}\right]$$

$$= \frac{A}{2}\left[\underbrace{e^{j\theta}\sum_{t=0}^{N-1} W_N^{t\overbrace{(1-f)}^{\text{call this } k}}}_{S_1} + \underbrace{e^{-j\theta}\sum_{t=0}^{N-1} W_N^{t(-f-1)}}_{S_2}\right]$$

— Re-write S_2 a bit : $S_2 = \sum_{t=0}^{N-1} W_N^{t(-f-1)} = \sum_{t=0}^{N-1} W_N^{t(-f-1)} \underbrace{W_N^{N}}_{1} = \sum_{t=0}^{N-1} W_N^{t\overbrace{(N-f-1)}^{\text{call this } l}}$

→ What are S_1 and S_2 ?

→ Same procedure:

$$S_1 = \sum_{t=0}^{N-1} W_N^{tk} = 1 + W_N^k + (W_N^k)^2 + \dots + (W_N^k)^{N-1}$$

$$W_N^k . S_1 = \quad = W_N^k + (W_N^k)^2 + \dots + (W_N^k)^{N-1} + (W_N^k)^N$$

$$S_1 - W_N^k S_1 = 1 - (W_N^k)^N \implies S_1 = \frac{1-(W_N^k)^N}{1-W_N^k}$$

→ if $k \neq 0, N, -N, \pm iN$, then $(1 - W_N^k)$ = denominator $\neq 0$

→ numerator: $1 - W_N^{kN} = 1 - (W_N^N)^k = 0$

→ $S_1 = 0$

→ if $k = 0, N, -N, \pm iN$, then $W_N^k = 1$

→ $S_1 = 1 + 1 + 1 \dots + 1 = N$

$$\implies S_1 = \begin{cases} N, & k=0, \pm iN \\ 0, & \text{otherwise.} \end{cases}$$

$\implies$ Now $k = 1-f$, $f = 0, \dots, N-1 \implies 1-f = 1, 0, \dots, -N+2$

$$\implies S_1 = \begin{cases} N, & \text{if } f=1 \\ 0, & \text{otherwise} \end{cases}$$

→ How about S_2? Same expression, replace k by ℓ.

$\ell = N-f-1$, $f = 0, \dots, N-1 \implies N-f-1 = N-1, N-2, \dots, \underbrace{N-(N-1)-1}_{0, \text{ when } f=N-1}$

$$\implies S_2 = \begin{cases} N, & \text{if } f=N-1 \\ 0, & \text{otherwise} \end{cases}$$

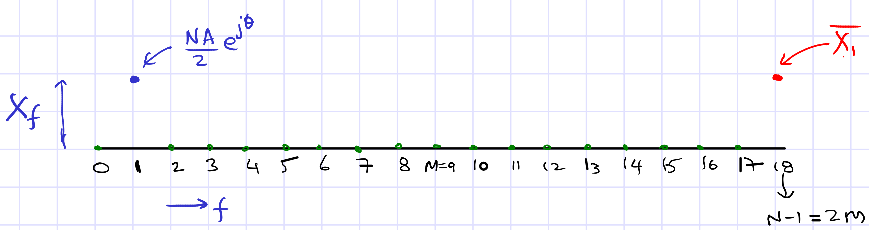
THEREFORE

→ ∴ we have:

→ $X_1 = N\frac{A}{2} e^{j\theta}$ (with $f=1 \implies S_1 = N$)

→ $X_{N-1} = N\frac{A}{2} e^{-j\theta} = \overline{X_1}$ (with $f=N-1 \implies S_2 = N$)

→ X_f for $f = 0, 2, 3, \dots, N-2 = 0$



→ SUMMARY : IF YOU DFT THE TIME-DOMAIN SAMPLES OF THE PHASOR $\frac{A}{2}e^{j\theta}$ (with $\omega = \frac{2\pi}{N}$)

→ X_1 IS SIMPLY THE PHASOR $\frac{A}{2}e^{j\theta}$ scaled by N

→ X_{N-1} is ITS CONJUGATE

→ THE REST ARE 0.

→ NEXT EXPERIMENTS (WORK OUT THE DETAILS YOURSELF!) ← DON'T SKIMP!

→ TAKE N TO BE AN ODD NUMBER: $N = 2M+1$

some positive integer

→ SUPPOSE YOU TRY $x(t) = A\cos\left(\frac{2\pi}{N} f^* t + \theta\right)$

f^* CAN TAKE INTEGER VALUES FROM 1 TO M

→ PHASOR FORM: $\frac{A}{2}e^{j\theta}$, with angular freq. $\omega = \frac{2\pi f^*}{N}$

→ SOME TERMINOLOGY:

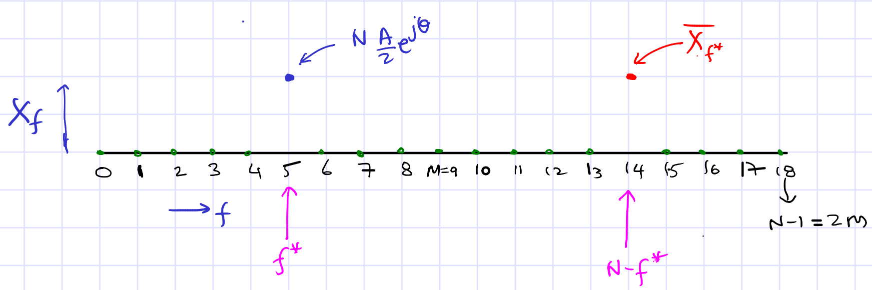
→ N = the time period (or just period)

→ $\frac{1}{N}$ = the fundamental frequency (or just fundamental)

→ $\frac{f^*}{N}$ = the $f^{*\,th}$ HARMONIC (of the fundamental)

→ SO $x(t)$ represents a sinusoid at the $f^{*\,th}$ harmonic with phasor $\frac{A}{2}e^{j\theta}$

→ THE DFT of $\vec{x}$ / $x[t]$ WILL LOOK LIKE THIS (PROVE YOURSELF - SIMILAR TO THE $f^*=1$ CASE ABOVE)



→ NOW SUPPOSE $x(t)$ = sum of DC + sinusoids at the fundamental + harmonics

→ i.e., $x(t) = x_{DC}$

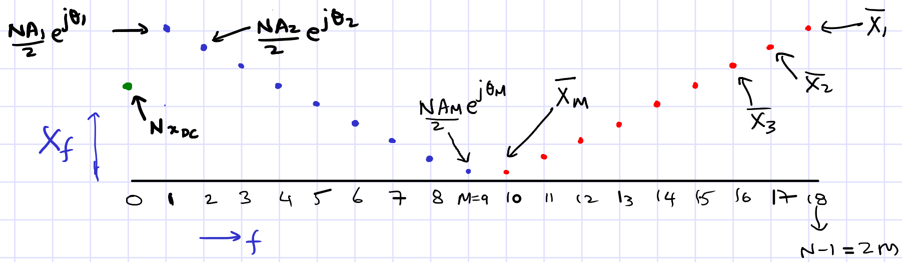
$+ \frac{A_1}{2}\cos\left(\frac{2\pi}{N}t + \theta_1\right)$ ← fundamental

$+ \frac{A_2}{2}\cos\left(\frac{2\pi}{N}\cdot 2\cdot t + \theta_2\right)$ ← 2^{nd} harmonic

$+ \cdots$

$+ \frac{A_M}{2}\cos\left(\frac{2\pi}{N}Mt + \theta_M\right)$ ← m^{th} harmonic

→ IF YOU RUN A DFT on $x[t]\,/\,\vec{x}$, you will get:



→ i.e, $X_0 = N x_{DC}$

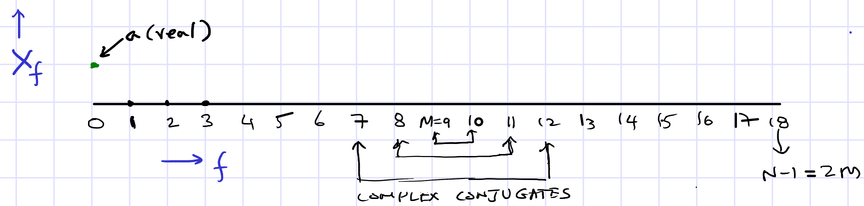
$$X_i = \frac{NA_i}{2} e^{j\theta_i}, \quad i = 1, \dots, M$$

and $X_{N-i} = \overline{X_i}$, $i = 1, \dots, M$.

→ IN OTHER WORDS: THE DFT DECOMPOSES $\vec{x}\,/\,x[t]$ INTO SINUSOIDS AT FREQUENCIES $0 = DC$, $\frac{2\pi}{N}$ = fundamental, $\frac{2\pi}{N}\cdot 2 = 2^{nd}$ harmonic, etc, upto $\frac{2\pi}{N}M = M^{th}$ harmonic.

→ FINDING THE PHASOR AT EACH FREQUENCY (SCALED BY N) AS X_i

→ Choose $\underline{X_0 = \overset{\text{REAL}}{a}}$, $X_f = 0$, $f = 1, 2, \cdots, N-1$



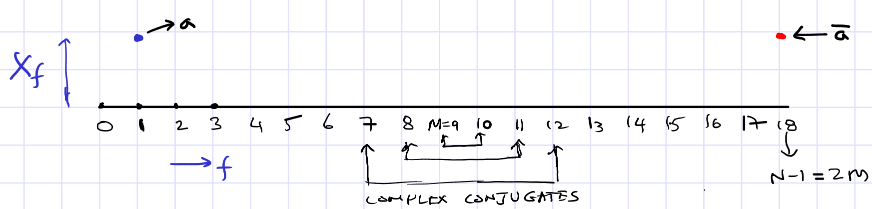
→ WHAT IS ITS IDFT?

$$\rightarrow \quad x_t = \frac{1}{N} \sum_{f=0}^{N-1} W_N^{tf} X_f = \frac{a}{N}, \quad \text{for } t = 0, 1, 2, \cdots, N-1$$

→ THEREFORE, X_0 CAPTURES A $\underline{\text{DC WAVEFORM}}$ (interpreting x_t as $x[t]$)

→ WHAT ABOUT X_1 ?

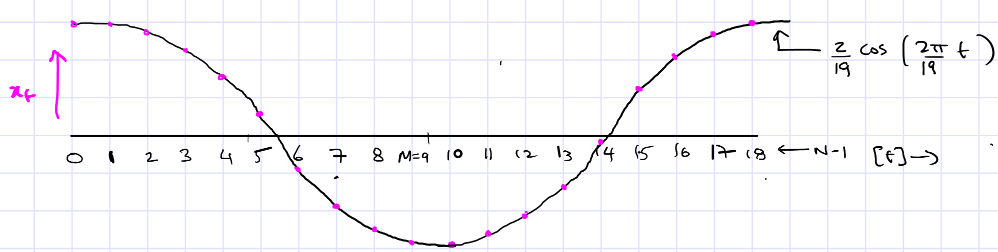
→ CHOOSE: $X_0 = 0$, $X_1 = \overset{\text{COMPLEX}}{a}$, $X_{N-1} = \bar{a}$, the rest $= 0$



$$\rightarrow \quad x_t = \frac{1}{N} \sum_{f=0}^{N-1} W_N^{tf} X_f = \frac{1}{N}\left(W_N^{t} a + W_N^{t(N-1)} \bar{a}\right) = \frac{1}{N}\left(W_N^{t} a + W_N^{-t} \bar{a}\right)$$

$$= \frac{2}{N} \operatorname{Re}\{W_N^{t} a\}$$

→ say $a = 1$ (for simplicity), then $\underline{x_t = \frac{2}{N} \cos\left(\frac{2\pi}{N} t\right)}$



→ THIS IS A (CO) SINE WAVE (DISCRETE SAMPLES OF IT)

→ SUPPOSE $X_1 = a$ (some general complex number), $X_{N-1} = \bar{a}$

→ consider $X_1 = a$ to be a PHASOR

→ and write it in magnitude-phase form: $a = \overset{\text{real}, >0}{\overset{\downarrow}{A}} e^{j\theta}$

→ then $x(t) = \frac{2}{N} \mathrm{Re}\{ w_N^t a \} = \frac{2}{N} \mathrm{Re}\{ w_N^t A e^{j\theta} \}$

$$= \frac{2A}{N} \mathrm{Re}\{ e^{j\frac{2\pi t}{N}} e^{j\theta} \} = \frac{2A}{N} \mathrm{Re}\{ e^{j(\frac{2\pi}{N}t + \theta)} \}$$

$$\Rightarrow x_t = \underbrace{\frac{2A}{N}}_{\text{Amplitude}} \cos\Big(\underbrace{\frac{2\pi}{N}}_{\text{frequency}} t + \underbrace{\theta}_{\text{phase shift.}} \Big)$$

→ compare against what a "regular phasor" means:

→ Given $a = Ae^{j\theta}$, the time-domain waveform is:

$$a e^{j\omega t} + \bar{a} e^{-j\omega t} = 2\,\mathrm{Re}\{ a e^{j\omega t} \} = \underline{2A \cos(\omega t + \theta)}$$

→ SO NOW WE SEE THAT IF we interpret X_1 as a phasor, x_t is simply the samples of the regular phasor's time-domain waveform:

→ scaled by $1/N$

→ with $\omega = \frac{2\pi}{N}$ ← THIS IS CALLED THE FUNDAMENTAL (ANGULAR) FREQUENCY

→ WHAT ABOUT $X_2, X_3, \cdots$?

→ LET'S BE A BIT MORE PRECISE.

→ TAKE N TO BE ODD ⇒ $N = 2\overset{\text{INTEGER} > 0}{\overset{\downarrow}{M}} + 1$

→ TAKE $X_i = a$, $X_{N-i} = \bar{a}$, for some $0 < i \leq M$

→ with $a = Ae^{j\theta}$, as before.

→ THEN, $x_t = \frac{1}{N} \sum_{f=0}^{N-1} w_N^{ft} X_f = \frac{1}{N}\left(w_N^{it} X_i + w_N^{(N-i)t} X_{N-i} \right) = \frac{1}{N}\left(w_N^{it} a + w_N^{-it} \bar{a} \right)$

$$= \frac{2}{N} \mathrm{Re}\{ w_N^{it} a \} = \underline{\frac{2A}{N} \cos\left(\frac{2\pi i}{N} t + \theta \right)}$$

→ IF WE INTERPRET X_i as a phasor, x_t = samples of the "regular phasor time domain waveform",

→ scaled by $\frac{1}{N}$

→ with $\omega = \frac{2\pi}{N} i$ ← i^{th} HARMONIC OF FUNDAMENTAL $= i\,(\overset{\text{FUNDAMENTAL}}{\overset{\downarrow}{\tfrac{2\pi}{N}}})$

→ SUMMARY

→ take N odd ⇒ $N = 2M+1$

→ $X_0 = a$ (real) ⟼ DC $\vec{x} = \frac{a}{N}$

→ $X_i = a$ (phasor) ⟼ sinusoidal $\vec{x}$: $x_t = \frac{2}{N}|a| \cos\left(\frac{2\pi i}{N} t + \theta\right)$ → angle of a, ie. θ

→ for $i = 1, \cdots, M$

→ $X_{N-i} = \bar{a}$ REQUIRED FOR $\vec{x}$ to be REAL.

→ POINT TO PONDER (AND WORK OUT ON YOUR OWN)

→ HOW DOES THE ABOVE CHANGE IF N IS EVEN ?

→ NOW WE SEE WHAT THE DFT DOES:

→ $\vec{X} = F_N \vec{x} \iff \vec{x} = \frac{1}{N} F_N^* \vec{X}$

→ $\vec{X}$ CONTAINS:

→ the DC component (X_0)

→ phasors with angular frequencies $\frac{2\pi i}{N}$, $i = 1, \cdots, M$

→ I.E., $\vec{x}$ (= $x[t]$) HAS BEEN DECOMPOSED BY THE DFT INTO A SUM OF $\underbrace{\text{THE ABOVE PHASORS WITH THE ABOVE FREQUENCIES}}$ + DC.

↑ DISCRETE SAMPLES OF SINUSOIDS WITH FREQUENCIES $\frac{2\pi i}{N}$ ← FUNDAMENTAL, $i=1,\dots,M$. AND HARMONICS.

→ THIS IS WHY $\vec{X}$ is CALLED THE FREQUENCY-DOMAIN REPRESENTATION OF $\vec{x}$.