THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

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1. DFTs: INTRODUCTION

→ THE DFT IS JUST A MATRIX (SQUARE MATRIX)

→ BUT A VERY SPECIAL MATRIX

→ FILLED WITH (SPECIAL) COMPLEX NUMBERS

→ INVERTIBLE (AND ITS INVERSE IS "ALMOST ITSELF")

→ WIDELY USEFUL

→ CONVERT "TIME-DOMAIN" TO "FREQUENCY DOMAIN" AND VICE-VERSA

→ DIGITAL SIGNAL PROCESSING

→ FIND FREQUENCY SPECTRUM ✓

→ PERFORM FILTERING ✓

→ FOURIER ANALYSIS / SPECTROGRAMS / ETC. ✓

→ IMAGE COMPRESSION

→ JPEG IMAGES (DCT)

→ FAST COMPUTATION : FFT   COOLEY – TUKEY

→ SHOW LIVE DEMO + SLIDES

→ PRELIMINARIES

→ GIVEN ANY COMPLEX MATRIX A

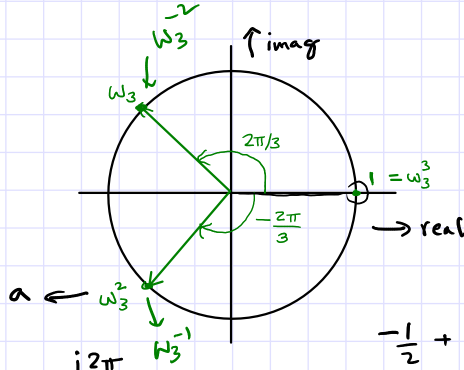
→ DENOTE BY A^* ITS COMPLEX CONJUGATE TRANSPOSE

→ $A^* = \overline{A^T} = (\overline{A})^T$

→ A^* (or A^H) is called the HERMITIAN of A.

→ $A = \begin{bmatrix} 1 & e^{j\pi} \\ 2 & 1+j \end{bmatrix} \Rightarrow A^* = \begin{bmatrix} 1 & 2 \\ e^{-j\pi} & 1-j \end{bmatrix}$

2. THE ROOTS OF 1



$\rightarrow \sqrt{1} = \pm 1$, FOCUS ON -1.

$\rightarrow \omega_3 = \sqrt[3]{1} \iff \omega_3^3 = 1$;

$-\frac{1}{2} + \frac{\sqrt{3}}{2} j = \omega_3$

$\omega_3 = e^{j\frac{2\pi}{3}}$

$\omega_3^3 = \left(e^{j\frac{2\pi}{3}}\right)^3 = e^{j2\pi} = 1$

$\rightarrow \sqrt[N]{1} = \underline{e^{\frac{j2\pi}{N}} \stackrel{\Delta}{=} \omega_N}$

$a = e^{-j\frac{2\pi}{3}} \Rightarrow a^3 = e^{-j2\pi} = 1$

$\rightarrow \omega_N^i$ is also a root of 1

$\omega_3 = e^{j\frac{2\pi}{3}}$

$\rightarrow$ for $i = 0, 1, 2, N-1$

$\omega_3^2 = e^{j\frac{4\pi}{3}} = e^{-j\frac{2\pi}{3}}$

$\rightarrow (\omega_N^i)^N = \omega_N^{iN} = (\omega_N^N)^i = 1^i = 1$

$\rightarrow$ COMPLEX CONJUGATE OF $e^{j\theta} = e^{-j\theta}$

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3. THE DFT MATRIX

$$\cancel{A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \qquad a_{11},\ a_{12},\ a_{21},\ a_{22}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad a_{00},\ a_{01},\ a_{10},\ a_{11}$$

$\rightarrow$ CONVENTION: ROW/COL NUMBERING STARTS FROM 0

$\rightarrow$ DFT MATRIX OF SIZE $\underline{N}$: F_N

$\rightarrow \underline{(i,j)^{th} \text{ entry} : \omega_N^{-ij}}$ $\qquad i, j = 0, \dots, N-1$

$\rightarrow$ EXAMPLE: $N = 3$

$\omega_N = \omega_3 = e^{j\frac{2\pi}{3}}$

$$\omega_3^{-4} = \omega_3^{-3-1} = \omega_3^{-3}\omega_3^{-1}, \qquad \omega_3^{-3} = \frac{1}{\omega_3^3}$$

$$F_3 = \begin{array}{c|c} j \rightarrow,\ i \downarrow & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \omega_3^{-0\times 0} & \omega_3^{-0\times 1} & \omega_3^{-0\times 2} \\ \omega_3^{-1\times 0} & \omega_3^{-1\times 1} & \omega_3^{-1\times 2} \\ \omega_3^{-2\times 0} & \omega_3^{-2\times 1} & \omega_3^{-2\times 2} \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3^{-1} & \omega_3^{-2} \\ 1 & \omega_3^{-2} & \omega_3^{-1} \end{bmatrix}$$

→ DFT × VECTOR PRODUCT FORMULA

→ say $\vec{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$, and define $\vec{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = DFT * \vec{x} = F_N \vec{x}$

→ "SIMPLE" FORMULA FOR y_i ?

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_i \\ \vdots \\ y_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} & & & \\ \omega_N^{-i\cdot 0} & \omega_N^{-i\cdot 1} & \cdots & \omega_N^{-i(N-1)} \\ \cdot & & & \end{bmatrix}}_{F_N} \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

jth row

$$\Longrightarrow \quad y_i = \sum_{j=0}^{N-1} \omega_N^{-ij} x_j$$

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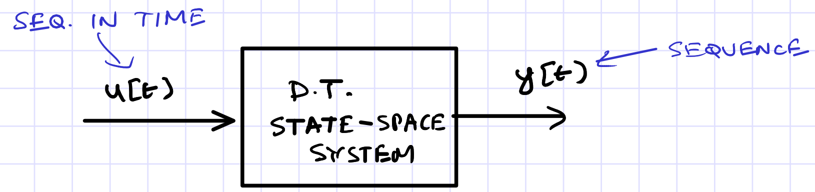
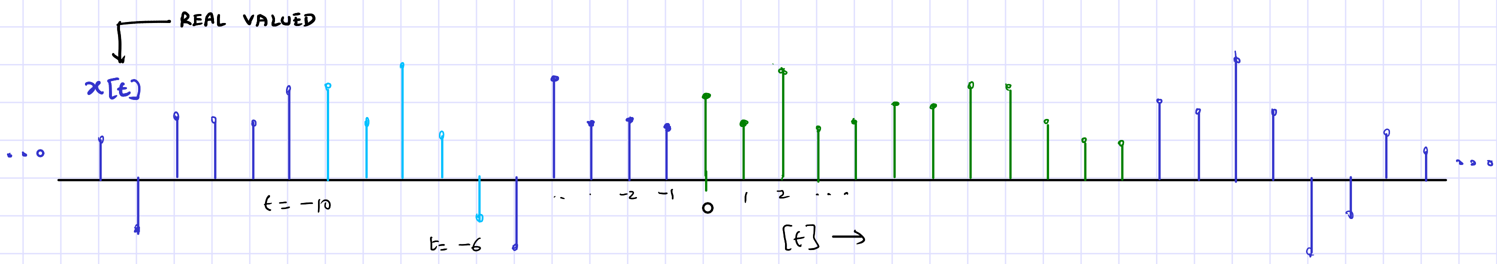
4. CHUNKS OF SEQUENCES INTERPRETED AS VECTORS



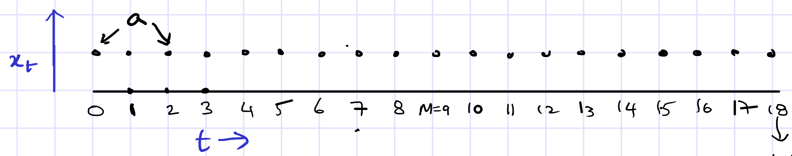
$N = 5$



$$\vec{x} = \begin{bmatrix} x[-10] \\ x[-9] \\ x[-8] \\ \vdots \\ x[-6] \end{bmatrix} \begin{matrix} \leftarrow x_0 \\ \leftarrow x_1 \\ \vdots \\ \\ \leftarrow x_5 \end{matrix}$$

5. FREQUENCY INTERPRETATION OF THE DFT

→ TRY TAKING THE DFT OF A CHUNK OF A "DC" SEQUENCE



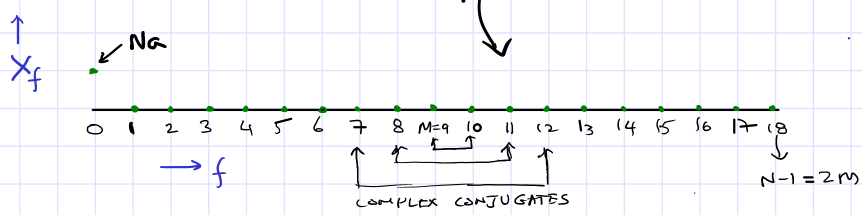
$x[t]$ constant with $t \in \{0, \dots, N-1\}$

⟹ DC

$$\vec{x} = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} \quad N=19$$

$$\vec{X} = F_N \vec{x}$$

$N-1 = 2M$



→ NOW TRY: $x(t) = \underbrace{A \cos\left(\frac{2\pi}{N} t + \theta\right)}$ $= \frac{A}{2}\left(e^{j\frac{2\pi}{N}t} e^{j\theta} + e^{-j\frac{2\pi}{N}t} e^{-j\theta}\right)$

C.T waveform

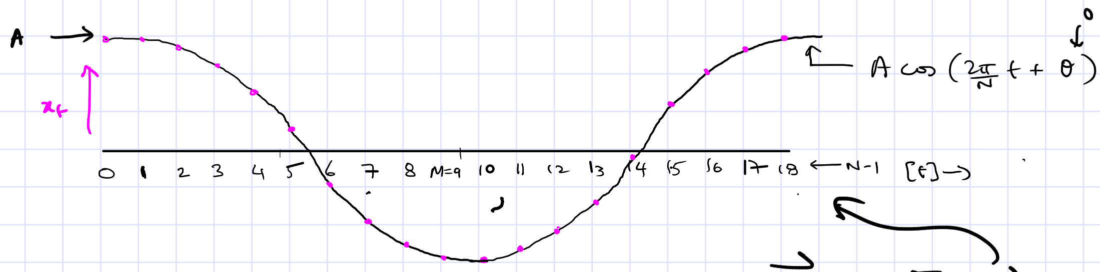
PHASOR: $\frac{A}{2} e^{j\theta}$ at freq. $\omega = \frac{2\pi}{N}$

→ RECALL: Given any phasor B (complex)

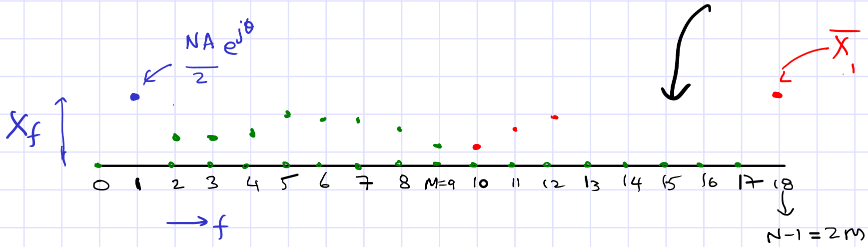
→ it MEANS a time-domain waveform

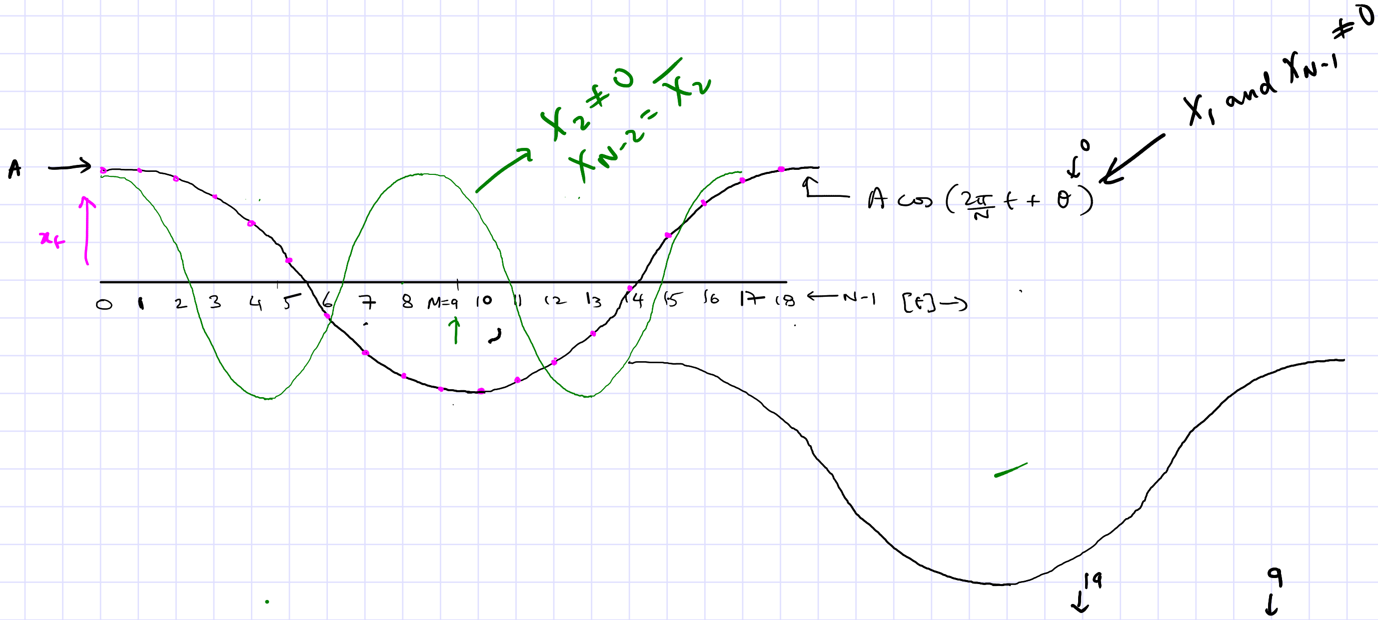
$$b(t) = B e^{j\omega t} + \bar{B} e^{-j\omega t}$$

→ OR, $x[t] = A \cos\left(\frac{2\pi}{N} t + \theta\right)$, t = INTEGERS ONLY



$$\vec{X} = F_N \vec{x}$$





$N = \text{odd} = 2M+1$ (19 = 2·9+1)

→ IN OTHER WORDS: THE DFT DECOMPOSES $\vec{x}$ / $x[t]$ INTO SINUSOIDS AT FREQUENCIES 0 = DC, $\frac{2\pi}{N}$ = fundamental,

$\frac{2\pi}{N}\cdot 2$ = 2nd harmonic, etc, upto

$\frac{2\pi}{N}M$ = Mth harmonic.

→ FINDING THE PHASOR AT EACH FREQUENCY (SCALED BY N) AS X_i

→ $\vec{X}$ = "FREQUENCY DOMAIN REPRESENTATION" OF $\vec{x}$

$A\cos(\omega t + \phi)$