

THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

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3. THE DFT MATRIX ✓
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5.

$$\text{DFT} : \vec{X} = F_N \vec{x}$$

$$N = \text{odd} = 2M + 1$$

$\downarrow$ $\downarrow$
 11 5

TIME DOMAIN

$$\text{DFT} = F_N$$

$$\text{IDFT} = \frac{1}{N} F_N^*$$

FREQUENCY DOMAIN

$$\vec{x}$$

$$\vec{X} = F_N \vec{x}$$

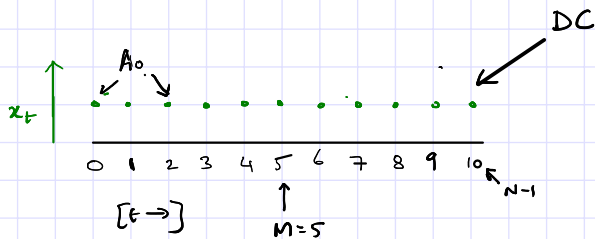
TD

DC SIGNAL

FD

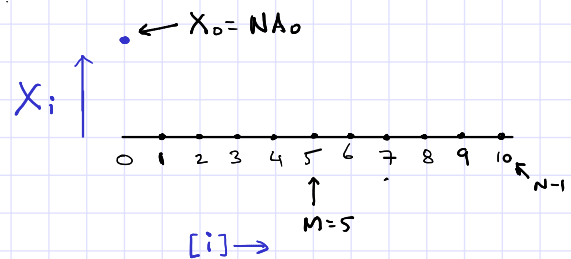
$$\rightarrow x(t) = A_0 \leftarrow \text{DC} \quad \text{REAL}$$

$$\rightarrow x_t = x[t] = a, \quad t = 0, \dots, N-1$$



$$\rightarrow X_0 = N A_0 \leftarrow \text{REAL}$$

$$\rightarrow X_1, X_2, \dots, X_{10} = 0$$



TD

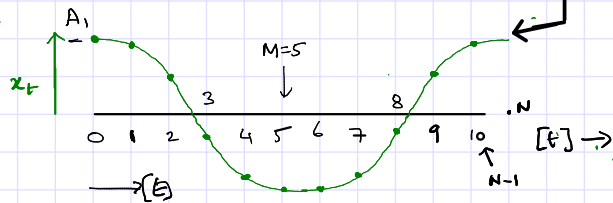
SINUSOID AT FUNDAMENTAL: FREQUENCY = $\frac{2\pi}{N}$ $\leftarrow$ ANGULAR FREQUENCY

FD

$$\rightarrow x(t) = A_1 \cos\left(\frac{2\pi}{N} t + \theta_1\right) \leftarrow \text{REAL}$$

$$f = \frac{1}{N}, T = \frac{1}{f} = N \rightarrow \text{PHASOR: } \frac{A_1}{2} e^{j\theta_1} \text{ at } \omega = \frac{2\pi}{N}$$

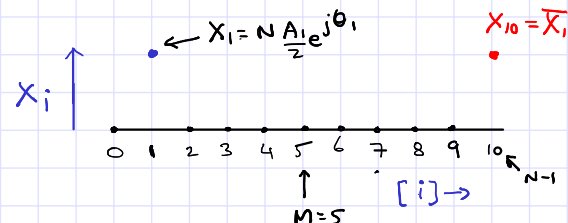
$$\rightarrow x_t = x(t), \quad t = 0, \dots, N-1$$



$$\rightarrow X_1 = \frac{N A_1}{2} e^{j\theta_1} \leftarrow \text{COMPLEX}$$

$$\rightarrow X_{N-1} = X_0 = \bar{X}_1$$

$$\rightarrow X_0, X_2, \dots, X_9 = 0$$



TD

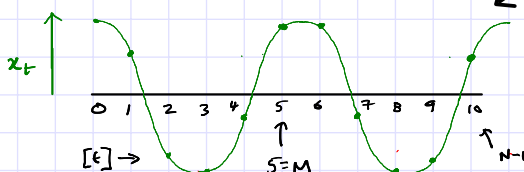
SINUSOID AT 2nd HARMONIC: FREQUENCY = $2 \times \frac{2\pi}{N}$

FD

$$\rightarrow x(t) = A_2 \cos\left(2 \frac{2\pi}{N} t + \theta_2\right) \leftarrow \text{REAL}$$

$$\rightarrow \text{PHASOR: } \frac{A_2}{2} e^{j\theta_2}, \text{ at } \omega = 2 \frac{2\pi}{N}$$

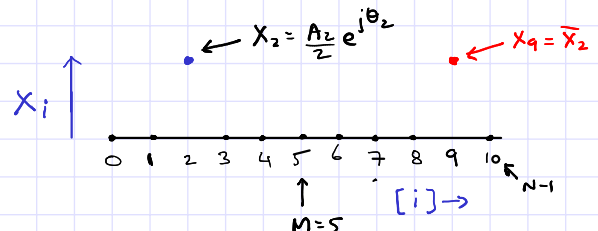
$$\rightarrow x_t = x(t), \quad t = 0, \dots, N-1$$



$$\rightarrow X_2 = \frac{A_2}{2} e^{j\theta_2} \leftarrow \text{COMPLEX}$$

$$\rightarrow X_{N-2} = X_9 = \bar{X}_2$$

$$\rightarrow X_0, X_1, X_3, \dots, X_8, X_{10} = 0$$



$$\begin{aligned}
 A_1 \cos(2\pi f_0 t + \theta_1) &= \frac{A_1}{2} \left(\underbrace{e^{j\theta_1}}_{\substack{\text{PHASOR} = B \\ \rightarrow e^{j(2\pi f_0 t + \theta_1)}}} e^{j2\pi f_0 t} + e^{-j\theta_1} e^{-j2\pi f_0 t} \right) \\
 &\rightarrow \cancel{2 \cos(2\pi f_0 t + \theta_1)} + \cancel{j \sin(2\pi f_0 t + \theta_1)} \\
 &\quad + \cancel{\cos(2\pi f_0 t + \theta_1)} - \cancel{j \sin(2\pi f_0 t + \theta_1)} \\
 &= \left(B e^{j\omega t} + \overline{B} e^{-j\omega t} \right)
 \end{aligned}$$

$$\begin{aligned}
 A_1 \cos(2\pi f_0 t + \theta_1) &= \frac{A_1}{2} \left(e^{j\theta_1} e^{j2\pi f_0 t} + e^{-j\theta_1} e^{-j2\pi f_0 t} \right) \\
 &= \underbrace{\frac{A_1}{2} e^{j\theta_1}}_{\text{PHASOR}} e^{j2\pi f_0 t} + \underbrace{\frac{A_1}{2} e^{-j\theta_1}}_{\text{PHASOR}} e^{-j2\pi f_0 t}
 \end{aligned}$$

SINUSOID AT 3rd HARMONIC: FREQUENCY = $3 \times \frac{2\pi}{N}$

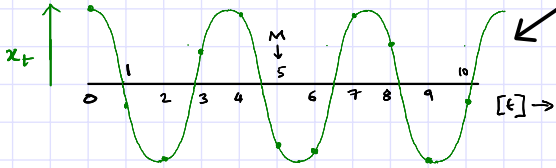
TD

FD

$$\rightarrow x(t) = A_3 \cos(3 \frac{2\pi}{N} t + \theta_3) \leftarrow \text{REAL}$$

$$\rightarrow \text{PHASOR: } \frac{A_3}{2} e^{j\theta_3}, \text{ at } \omega = 3 \frac{2\pi}{N}$$

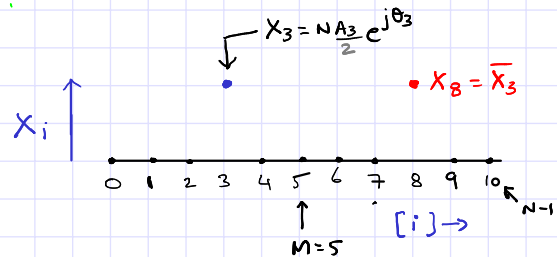
$$\rightarrow x_t = x(t), \quad t=0, \dots, N-1$$



$$\rightarrow X_3 = \frac{A_3}{2} e^{j\theta_3} \leftarrow \text{COMPLEX}$$

$$\rightarrow X_{N-3} = X_8 = \overline{X_3}$$

$$\rightarrow X_0, X_1, X_2, X_4, X_5, X_6, X_7, X_9, X_{10} = 0$$



SINUSOID AT Mth HARMONIC: FREQUENCY = $M \times \frac{2\pi}{N}$

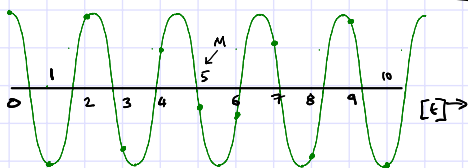
TD

FD

$$\rightarrow x(t) = A_M \cos(M \frac{2\pi}{N} t + \theta_M) \leftarrow \text{REAL}$$

$$\rightarrow \text{PHASOR: } \frac{A_M}{2} e^{j\theta_M}, \text{ at } \omega = \frac{2\pi}{N}$$

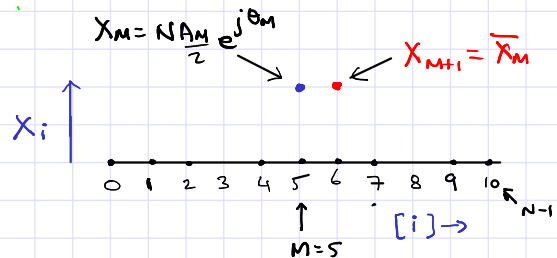
$$\rightarrow x_t = x(t), \quad t=0, \dots, N-1$$



$$\rightarrow X_M = \frac{A_M}{2} e^{j\theta_M} \leftarrow \text{COMPLEX}$$

$$\rightarrow X_{N-M} = X_{M+1} = \overline{X_M}$$

$$\rightarrow X_0, X_1, X_2, \dots, X_{M-1}, X_{M+2}, \dots, X_{N-1} = 0$$



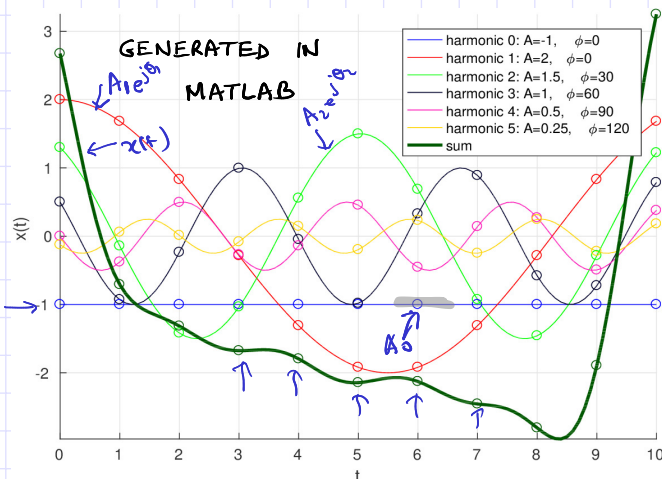
SUM OF ALL THESE SINUSOIDS AT HARMONICALLY RELATED FREQUENCIES (+ DC)

TD

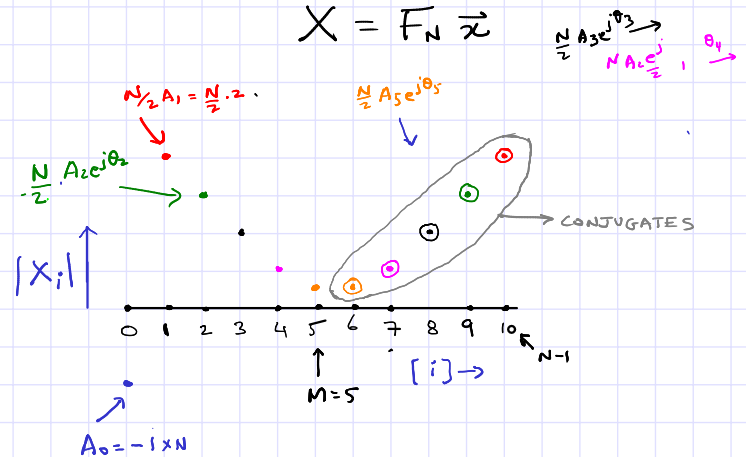
FD

$$x(t) = A_0 + \sum_{i=1}^M A_i \cos(\frac{2\pi i}{N} t + \theta_i) \leftarrow \text{REAL}$$

$$x_t = x(t), \quad t=0, \dots, N-1$$



$$\vec{X} = F_N \vec{x}$$



→ KEY POINT

→ TAKE ANY odd $N > 1$, $N = 2M + 1$

→ SOMEONE GIVES YOU N SAMPLES OF $x(t) = A_0 + \sum_{i=1}^M A_i \cos\left(\frac{2\pi i}{N}t + \theta_i\right)$ at $t=0, 1, \dots, N-1$

→ WITHOUT TELLING YOU A_0, A_i, θ_i — i.e., YOU DON'T KNOW THE PHASORS $\frac{A_i e^{j\theta_i}}{2}$, OR A_0

→ JUST TAKE A DFT: $X = F_N \tilde{x}$ VECTOR OF THE SAMPLES

→ $\frac{1}{N} X_i$ ARE YOUR PHASORS ($i=1, \dots, M$); $\frac{1}{N} X_0$ IS THE DC TERM!

→ ALL YOU NEED ARE THE SAMPLES: THE DFT GETS YOU THE PHASORS IMMEDIATELY

→ CHANGING THE TIME PERIOD / FUNDAMENTAL FREQUENCY

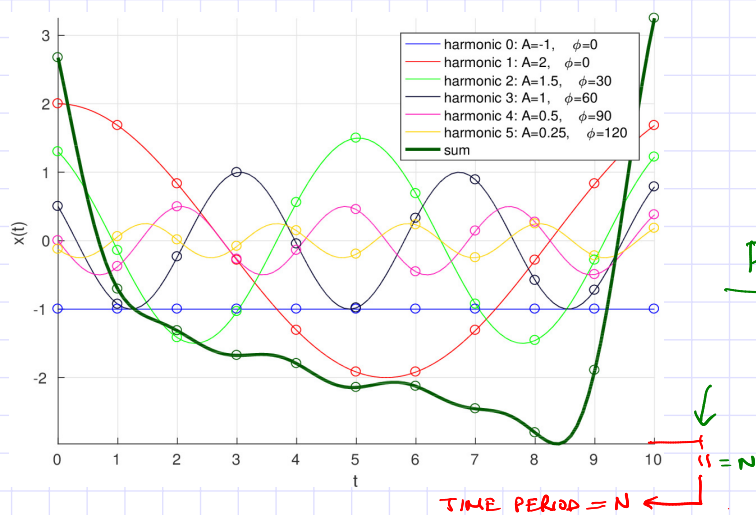
→ We had $x(t) = A_0 + \sum_{i=1}^M A_i \cos\left(\frac{2\pi i}{N}t + \theta_i\right)$

$2\pi f_0 i$
 $\downarrow$
 $\frac{1}{N}$



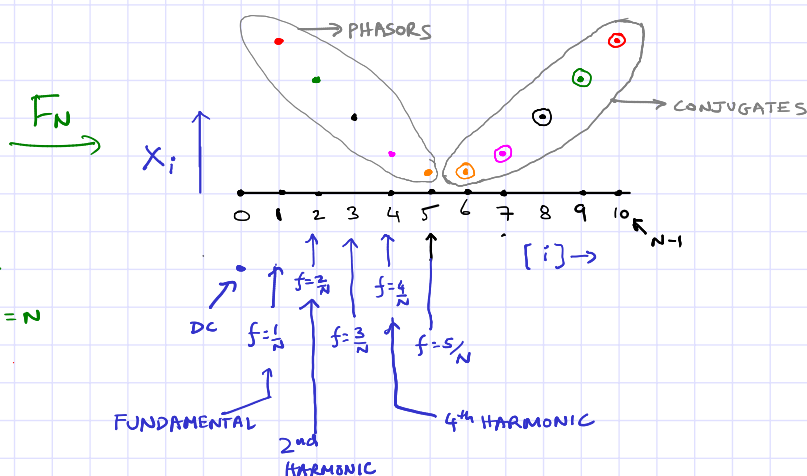
→ THE TIME PERIOD IS N ← RECALL: $\cos\left(\frac{2\pi}{T}t\right)$ has a period of T .

→ FUNDAMENTAL FREQ = $\frac{1}{\text{TIME PERIOD}} = \frac{1}{N} = f_0$



frequency of i th harmonic
 $\downarrow$
 $i f_0$

$$x(t) = A_0 + \sum_{f=1}^M A_f \cos\left(2\pi \left(\frac{i}{N}\right)t + \theta_i\right)$$



→ SUPPOSE WE WANT A DIFFERENT FUNDAMENTAL FREQUENCY

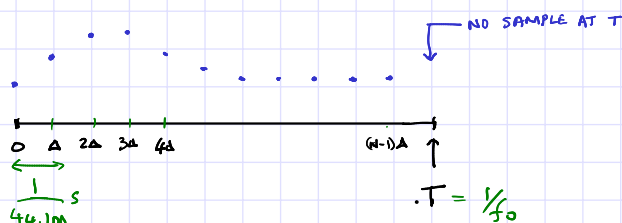
→ $x(t) = A_0 + \sum_{i=1}^M A_i \cos(2\pi f_0 i T + \theta_i)$ anything you like

⇒ TIME-PERIOD $T = \frac{1}{f_0}$

~~$t = 0, 1, 2, 3, \dots, N-1$~~

$t = 0, 0, 2\Delta, 3\Delta, \dots, (N-1)\Delta$

→ I STILL TAKE $N = 2M+1$ SAMPLES, SPACED AT $\Delta = \frac{T}{N} = \frac{1}{f_0 N}$



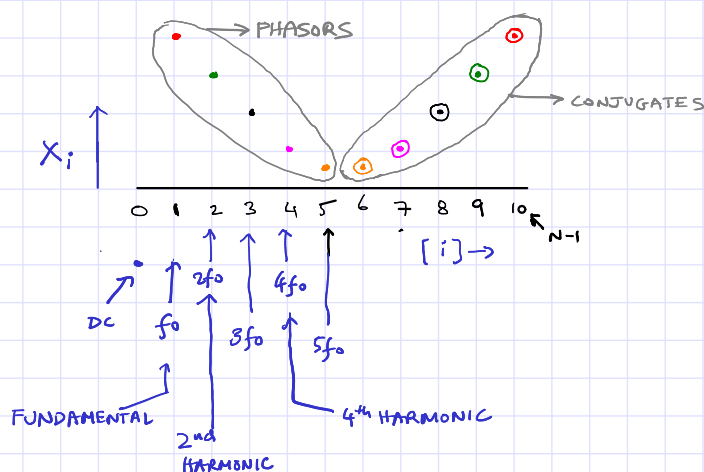
→ $\vec{x} = \begin{bmatrix} x(0) \\ x(\Delta) \\ x(2\Delta) \\ \vdots \\ x((N-1)\Delta) \end{bmatrix}$ ← x_0
← x_1
← x_{N-1}

→ IS THE DFT OF $\vec{x}$ STILL USEFUL?

→ YES: IT STILL GIVES YOU THE PHASORS
 → THE FUNDAMENTAL AND HARMONIC FREQUENCIES ARE DIFFERENT

→ $f_0, 2f_0, \dots, Mf_0$

→ INSTEAD OF $\frac{1}{N}, \frac{2}{N}, \dots, \frac{M}{N}$



DEMO: CALLING ELVIS IN MATLAB

→ CD FORMAT: SAMPLE RATE = 44,100 SAMPLES PER SECOND

⇒ $\Delta = \frac{1}{44,100}$ s ← SPACING OF TD SAMPLES

→ WE'LL TAKE 1 minute = 60 s OF AUDIO DATA

= $60 \times 44,100 = 2,646,000$ SAMPLES

→ WE'LL TAKE $N = 2,646,001$ (TO MAKE IT ODD) $N = 2M+1$

⇒ $M = \frac{N-1}{2} = 1,323,000$

$$\rightarrow T = N \cdot \Delta \Rightarrow T = \frac{2,646,001}{44,100} = 60 + 2.267 \times 10^{-5} \approx 60 \text{ s.}$$

$$N \log_2 N$$

$$(2.6 \times 10^6)^2 \approx 6 \times 10^{12}$$

$$\rightarrow \text{FUNDAMENTAL FREQ } f_0 = \frac{1}{T} = 1.667 \times 10^{-2} \text{ Hz. } \sim \frac{1}{60}$$

$$\rightarrow \text{HIGHEST HARMONIC} = M f_0 = \frac{M}{T} = \frac{M}{N \Delta} = \frac{M}{N} \times 44,100 = \frac{M}{2M+1} \cdot 44,100 \approx 22,050 \text{ Hz}$$

$$\rightarrow \text{DFT SIZE: } N \times N \leftarrow$$

$$\rightarrow F_N \vec{x}: \text{HOW MANY MULTIPLICATION OPERATIONS?}$$

$$\frac{1}{4 \times 10^9} \text{ s}$$

is

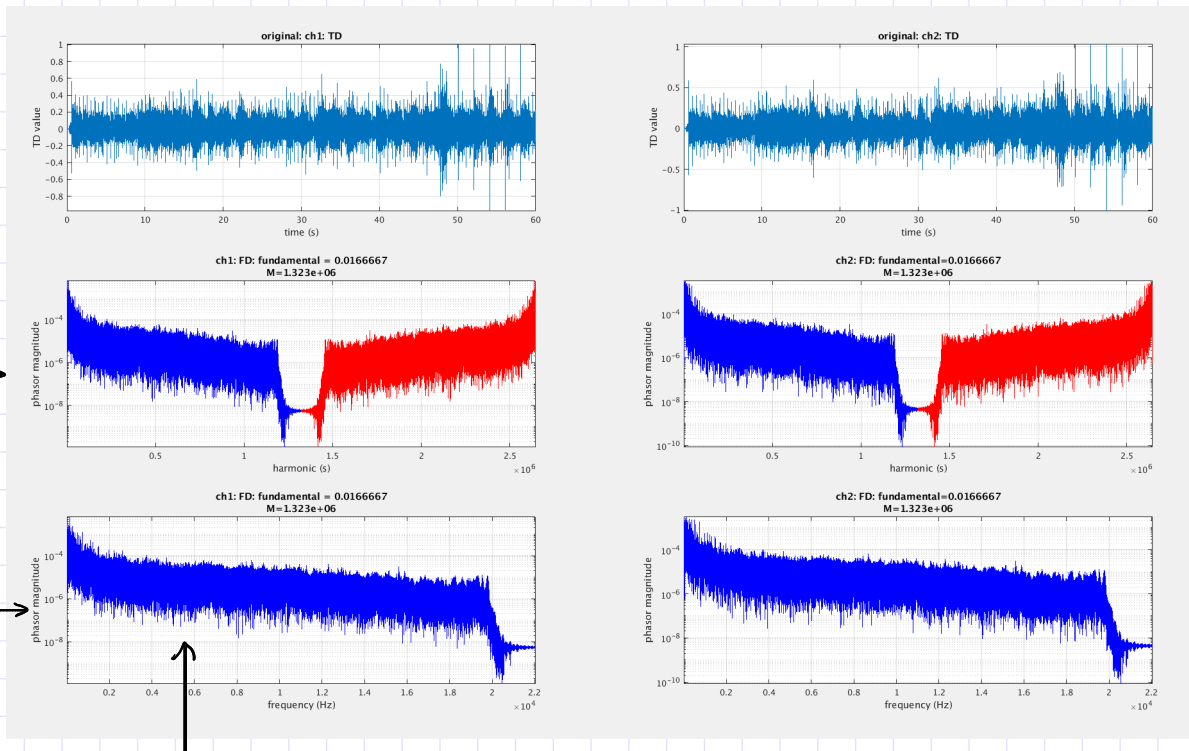
$$\rightarrow \text{MY COMPUTER'S CLOCK PERIOD: } \sim \frac{1}{4 \text{ GHz}} = 0.25 \times 10^{-9} \text{ s} \leftarrow \text{MINIMUM TIME NEEDED FOR ANY OPERATION}$$

$$\rightarrow \text{COMPUTER TIME NEEDED FOR } F_N \vec{x}: 6 \times 10^{12} \times 0.25 \times 10^{-9} = 1.5 \times 10^3 \text{ s}$$

$\rightarrow$ RUN MATLAB CODE calling `vis.m`.

$$\frac{x_i}{2} = \frac{A_i}{2} e^{j\theta_i} \rightarrow$$

$$X(f), f = if_0 \rightarrow i = 0, \dots, M$$



FREQUENCY SPECTRUM OF THE AUDIO SIGNAL

$\rightarrow$ RECALL (?) THE INVERSE DFT (IDFT)

$$\rightarrow \vec{X} = \underline{F_N} \vec{x} \iff \vec{x} = \underline{F_N^{-1}} \vec{X}$$

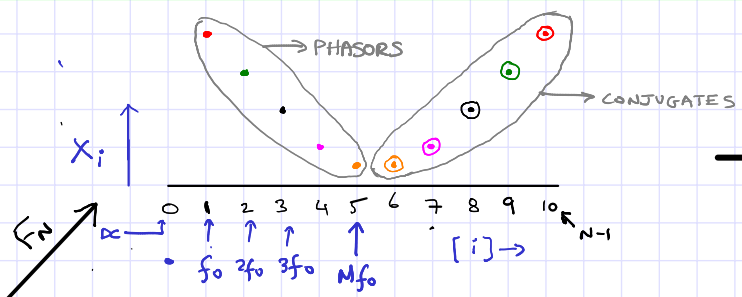
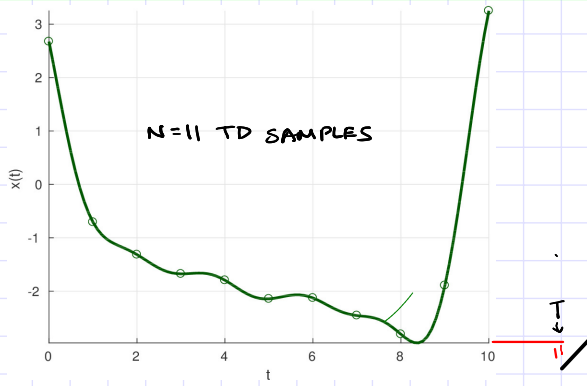
$$\rightarrow \text{CAN SHOW (see notes) that } \underline{F_N^{-1}} = \frac{1}{N} \underline{F_N}^*$$

INTERPOLATION VIA THE DFT

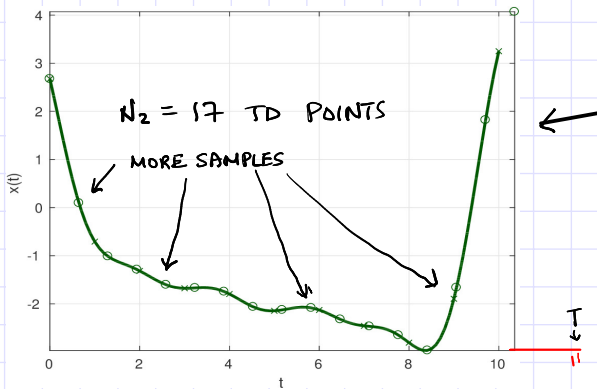
TD

$$x(t) = A_0 + \sum_{i=1}^M A_i \cos(2\pi f_0 i t + \theta_i) \quad \rightarrow 1/N$$

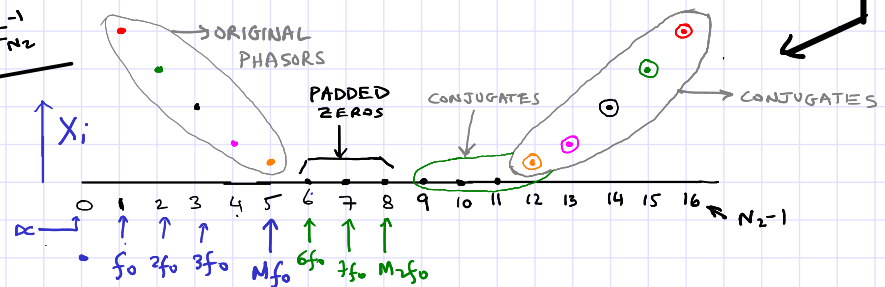
FD



→ INCREASE M IN THE F.D. REPRESENTATION:
→ $M_2 = (\text{say}) 8 \Rightarrow N_2 = 2M_2 + 1 = 17$



→ DON'T CHANGE THE SPECTRUM: JUST PAD WITH ZEROS



→ f_0 (FUNDAMENTAL) REMAINS THE SAME
→ SPECTRUM REMAINS THE SAME

$$x(t) = A_0 + \sum_{i=1}^{M_2} A_f \cos(2\pi f_0 i t + \theta_i)$$

→ with $A_6 = A_7 = A_8 = 0$: NO CHANGE TO $x(t)$

MORE SAMPLES VIA

"BAND LIMITED"

INTERPOLATION

→ DFT-BASED (BAND-LIMITED) INTERPOLATION: THE PROCEDURE

→ START with $\vec{x} \in \mathbb{R}^N$, $N = 2M + 1$ ← representing samples at $0, \frac{T}{N}, \frac{2T}{N}, \dots, \frac{(N-1)T}{N}$

→ $\vec{X} = F_N \vec{x}$ (SIZE N DFT)

→ CHOOSE SOME $M_2 > M$; DEFINE $N_2 = 2M_2 + 1$

→ DEFINE $\vec{X}_2$ BY:

→ $\vec{X}_2[0, \dots, M] = \frac{N_2}{N} \vec{X}[0, \dots, M]$ ← COPY THE PHASORS OF THE ORIGINAL HARMONICS

→ $\vec{X}_2[N_2 - M, \dots, N_2 - 1] = \frac{N_2}{N} \vec{X}[M + 1, \dots, N - 1]$ ← COPY THE CONJUGATES TO THE RIGHT PLACES

→ $\vec{X}_2[M + 1, \dots, N_2 - M - 1] = 0$ ← PAD THE EXTRA HARMONICS TO ZEROS.

→ $\vec{x}_2 = F_{N_2}^{-1} \vec{X}_2$ (SIZE N_2 INVERSE DFT)

↑ INTERPOLATED SAMPLES at $0, \frac{T}{N_2}, \frac{2T}{N_2}, \dots, \frac{(N_2-1)T}{N_2}$

$\frac{T}{N_2}, \frac{2T}{N_2}$

$$x(t) = A_0 +$$

$$A_1 \cos(2\pi f_1 t + \theta_1) +$$

$$A_2 \cos(2\pi f_2 t + \theta_2) +$$

⋮

$$A_5 \cos(2\pi f_5 t + \theta_5) +$$

$$A_6 \rightarrow 0 \cos(2\pi f_6 t + \theta_6) +$$

$$A_7 \rightarrow 0 \quad " \quad 7 \quad +$$

$$A_8 \rightarrow 0 \quad " \quad 8$$