INTERPOLATION

1. MOTIVATION: AUDIO TRANSMISSION EXAMPLE : SAMPLING AND INTERPOLATION
2. PWL AND ZOH INTERPOLATION
3. INTERPOLATION USING BASIS FUNCTIONS
4. SINC INTERPOLATION
5. INTERPOLATION USING (GLOBAL) POLYNOMIALS

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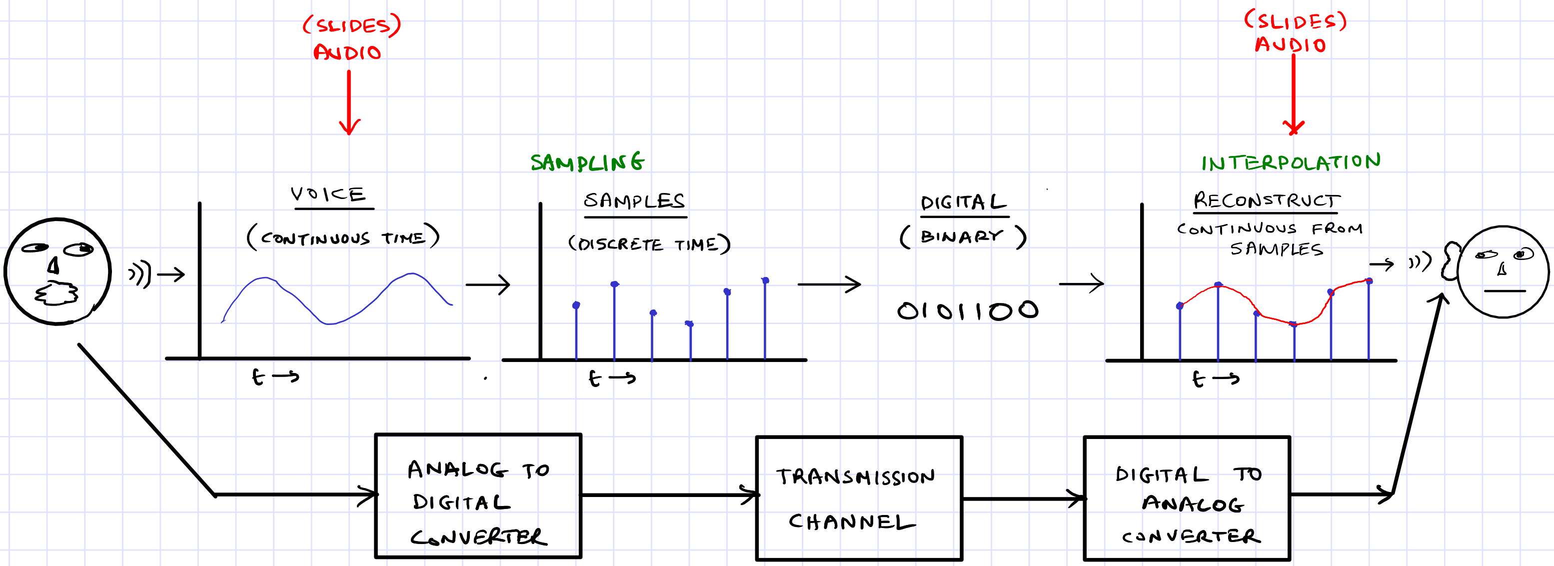
1. MOTIVATION



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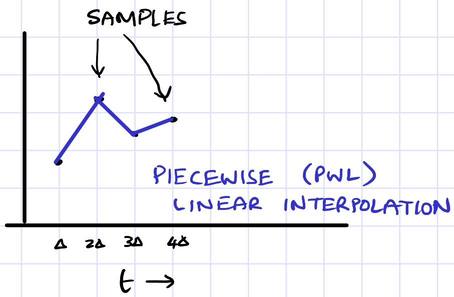
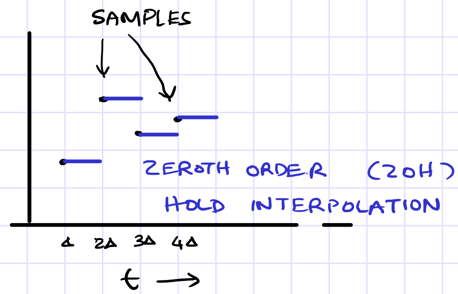
2. PIECEWISE LINEAR (PWL) AND ZEROTH-ORDER HOLD INTERPOLATION

→ INTERPOLATION: DEVISE "REASONABLE" VALUES BETWEEN DISCRETE SAMPLES.





→ STRAIGHT LINE BETWEEN ADJACENT POINTS
  → CONTINUOUS, NOT DIFFERENTIABLE
    → KINKS

→ HOLD PREVIOUS VALUE
  → NOT CONTINUOUS
    → JUMPS

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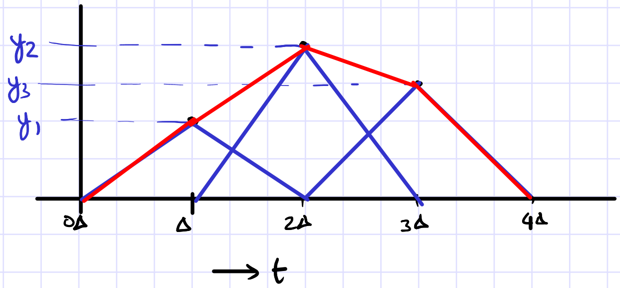
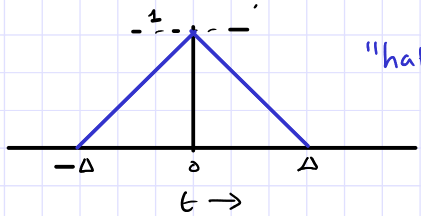
3. INTERPOLATION USING BASIS FUNCTIONS



Define $\phi(t) =$



"hat" function.

$$\phi(t) = \begin{cases} 0 & , \quad t < -\Delta \\ 1 + \frac{t}{\Delta} & , \quad -\Delta \le t \le 0 \\ 1 - \frac{t}{\Delta} & , \quad 0 \le t \le \Delta \\ 0 & , \quad t > \Delta \end{cases}$$

$$y(t) = \underset{\text{SCALE}}{y_1}\ \phi(t - \underbrace{1 \cdot \Delta}_{\text{TIME SHIFT}}) + y_2\, \phi(t - 2\cdot\Delta) + y_3\, \phi(t - 3\cdot\Delta)$$

THIS IS PWL INTERPOLATION

IN GENERAL: GIVEN SAMPLES $(t_0, y_0), (t_1, y_1), \ldots (t_{N-1}, y_{N-1})$

$\rightarrow$ $\{t_i\}$ equally spaced at Δ.

$$y_{PWL}(t) = \sum_{i=0}^{N-1} y_i\, \underbrace{\phi(t - t_i)}_{\text{BASIS FUNCTIONS}}$$

BASIS FUNCTIONS $\hookrightarrow$ LINEARLY INDEPENDENT

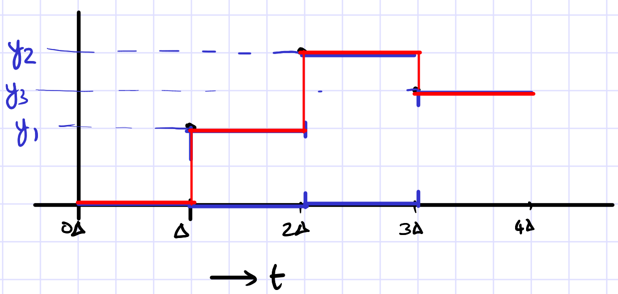
$\rightarrow$ WHY IS THE ABOVE WAY OF WRITING PWL INTERPOLATION POWERFUL?

$\rightarrow$ BECAUSE YOU CAN CHOOSE DIFFERENT $\phi(x)$ TO GET DIFFERENT INTERPOLATION SCHEMES.

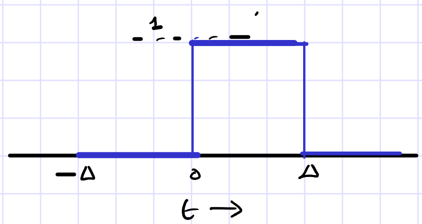
$\rightarrow$ ONLY REQUIREMENTS FOR $\phi(x)$:

1. $\phi(0) = 1$ $\leftarrow$ ENSURES $y_i\, \phi(t - t_i)$ passes through (t_i, y_i)
2. $\phi(k\Delta) = 0,\ k \neq 0$ $\leftarrow$ ENSURES $y_j\, \phi(t - t_j)$ does not affect (t_i, y_i), $j \neq i$

$\rightarrow$ CAN WE EXPRESS ZOH IN THIS FORM? WHAT IS $\phi(x)$ FOR ZOH?



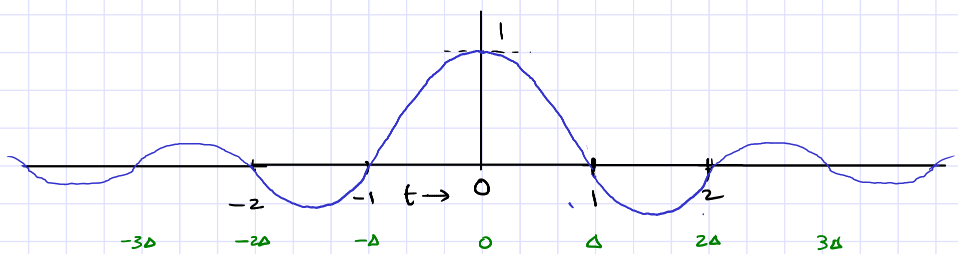
$\phi(t) =$



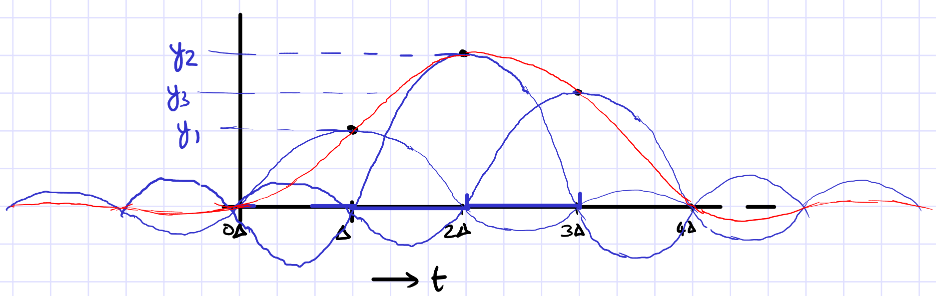
$\longrightarrow$ PWL VS. ZOH : RE-PLAY AUDIO (SLIDES)

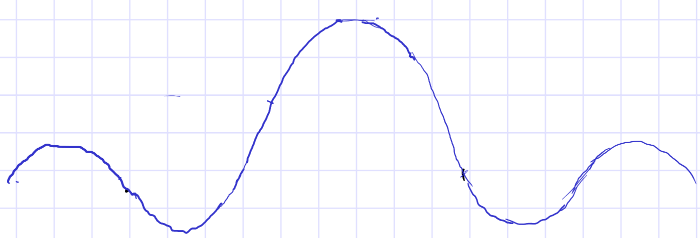
4. SINC INTERPOLATION

→ THE SINC FUNCTION: $\text{sinc}(t) \triangleq \begin{cases} 1, & t=0 \\ \dfrac{\sin(\pi t)}{\pi t}, & t \neq 0 \end{cases}$



→ CHOOSE $\phi(t) = \text{sinc}(t/\Delta)$ → then the zero-crossings above are at $k\Delta$





— ADVANTAGES (OVER PWL and ZOH):

— SMOOTH: NO KINKS OR JUMPS
— ALL DERIVATIVES CONTINUOUS

— BAND-LIMITED (WE'LL SEE LATER) ← IMPORTANT IN SIGNAL PROCESSING

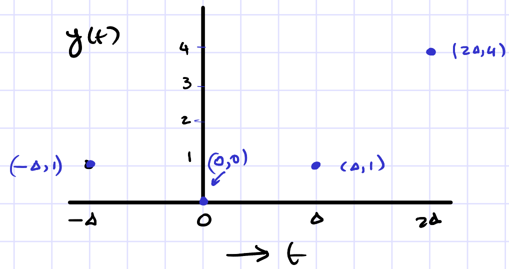
WHEN WE DO DFTs

→ EXAMPLES (SLIDES)

→ AUDIO
→ MICHEL'S PICTURE

5. INTERPOLATION USING GLOBAL POLYNOMIALS

→ ANOTHER WAY OF INTERPOLATION: FIT A POLYNOMIAL TO THE DATA



→ $N=4$ points: $(x_0, y_0), \cdots (x_3, y_3)$

→ FIT $(N-1)^{th} = 3^{rd}$ DEGREE POLYNOMIAL.

$$y(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 \rightarrow 4 \text{ unknowns}: a_0 \cdots a_3$$

— HOW TO SOLVE? JUST EVALUATE THE POLYNOMIAL at (x_i, y_i):

$$\left.\begin{aligned} y_0 &= a_3 t_0 + a_2 t_0^2 + a_1 t_0 + a_0 \\ y_1 &= a_3 t_1 + a_2 t_1^2 + a_1 t_1 + a_0 \\ y_2 &= a_3 t_2 + a_2 t_2^2 + a_1 t_2 + a_0 \\ y_3 &= a_3 t_3 + a_2 t_3^2 + a_1 t_3 + a_0 \end{aligned}\right] \Rightarrow \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\vec{a}}$$

Van der Monde matrix

$$\Rightarrow \quad V\vec{a} = \vec{y}$$

$$\Rightarrow \quad \vec{a} = V^{-1}\vec{y} \quad : \quad \text{IF } V^{-1} \text{ exists!}$$

→ PROPERTY OF VANDERMONDE MATRICES:

→ $\det V = \prod_{0 \le i < j \le N-1} (t_j - t_i)$

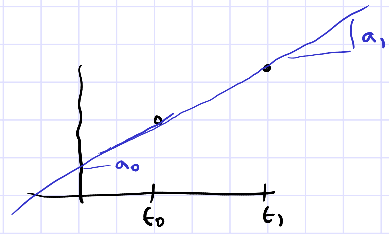
→ Always $\neq 0$ if $\{t_0, t_1, \cdots, t_{N-1}\}$ are all distinct.

→ FOR N POINTS $(t_0, y_0) \cdots (t_{N-1}, y_{N-1})$

$$V = \begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^{N-1} \\ 1 & t_1 & t_1^2 & & t_1^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_{N-1} & t_{N-1}^2 & & t_{N-1}^{N-1} \end{bmatrix}$$

— 2x2 EXAMPLE:

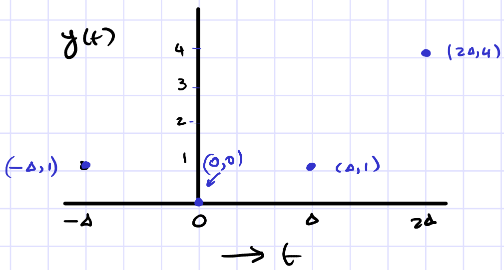
$\rightarrow$ $V = \begin{bmatrix} 1 & t_0 \\ 1 & t_1 \end{bmatrix} \Rightarrow V^{-1} = \frac{1}{t_1 - t_0}\begin{bmatrix} t_1 & -t_0 \\ -1 & 1 \end{bmatrix}$



$\rightarrow$ WHAT INTERPOLATION PROBLEM WILL THIS SOLVE?

$$y(t) = a_0 + a_1 t$$

$\rightarrow$ OUR INITIAL EXAMPLE IS EASY ENOUGH TO SOLVE BY INSPECTION:



$$y(t) = t^2 = 0 \cdot t^3 + 1 \cdot t^2 + 0 \cdot t + 0$$

$\rightarrow$ ADVANTAGES (OVER BASIS FUNCTION BASED INTERPOLATION)

$\rightarrow$ DON'T NEED TO ASSUME (ZERO) VALUES FOR $t = k\Delta$ beyond provided samples.

$\rightarrow$ (ALWAYS SMOOTH)

$\rightarrow$ DISADVANTAGES

$\rightarrow$ CAN GO WILD BETWEEN SAMPLES AND OUTSIDE

$\rightarrow$ EXAMPLES $\rightarrow$ SLIDES