THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

1. INTRODUCTION AND PRELIMINARIES ✓
2. ROOTS OF 1 ON THE COMPLEX PLANE ✓
3. THE DFT MATRIX ✓
4. VECTORS FROM SEQUENCES ✓
5. FREQUENCY INTERPRETATION OF DFT SIGNALS — WITHOUT PROOFS
6. DEMO: MATLAB
7. INTERPOLATION VIA THE DFT

5.

DFT : $\vec{X} = F_N \vec{x}$

$N = \text{odd} = 2M+1$

$N \downarrow 11$, $M \downarrow 5$

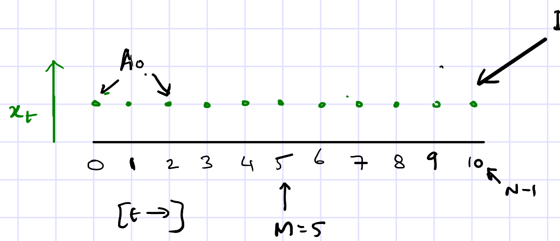
TIME DOMAIN	$\xrightarrow[\;IDFT = \frac{1}{N}F_N^*\;]{DFT = F_N}$	FREQUENCY DOMAIN
$\vec{x}$		$\vec{X}$

DC SIGNAL

TD

$\rightarrow x(t) = A_0$: DC ← REAL

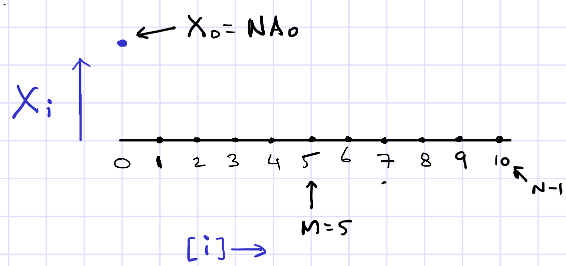
$\rightarrow x_t = x[t] = a, \quad t = 0, \dots, N-1$



FD

$\rightarrow X_0 = Na$ ← REAL

$\rightarrow X_1, X_2, \dots, X_{10} = 0$



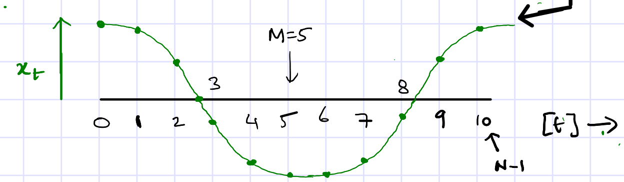
SINUSOID AT FUNDAMENTAL: FREQUENCY $= \frac{2\pi}{N}$

TD

$\rightarrow x(t) = A_1 \cos\left(\frac{2\pi}{N}t + \theta_1\right)$ ← REAL

$\rightarrow$ PHASOR: $\frac{A_1}{2}e^{j\theta_1}$ at $\omega = \frac{2\pi}{N}$

$\rightarrow x_t = x(t), \quad t = 0, \dots, N-1$

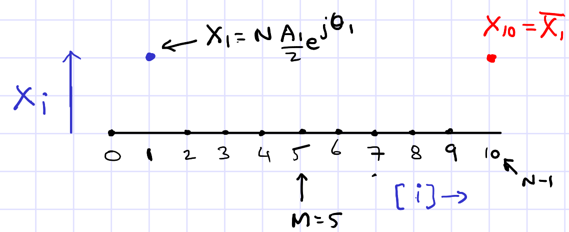


FD

$\rightarrow X_1 = N\frac{A_1}{2}e^{j\theta_1}$ ← COMPLEX

$\rightarrow X_{N-1} = X_{10} = \overline{X_1}$

$\rightarrow X_0, X_2, \dots, X_9 = 0$



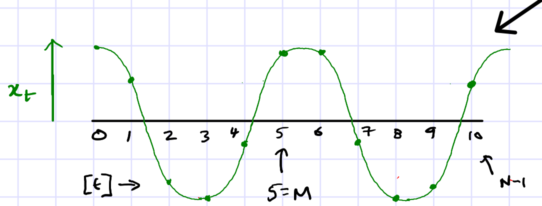
SINUSOID AT 2^{nd} HARMONIC: FREQUENCY $= 2 \times \frac{2\pi}{N}$

TD

$\rightarrow x(t) = A_2 \cos\left(2\,\frac{2\pi}{N}t + \theta_2\right)$ ← REAL

$\rightarrow$ PHASOR: $\frac{A_2}{2}e^{j\theta_2}$, at $\omega = 2\,\frac{2\pi}{N}$

$\rightarrow x_t = x(t), \quad t = 0, \dots, N-1$

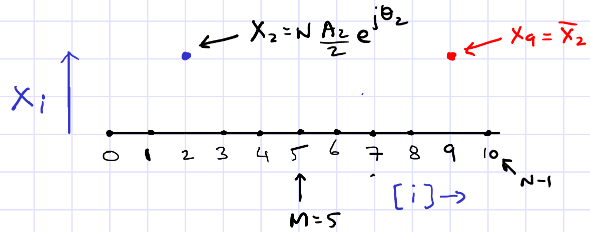


FD

$\rightarrow X_2 = N\frac{A_2}{2}e^{j\theta_2}$ ← COMPLEX

$\rightarrow X_{N-2} = X_9 = \overline{X_2}$

$\rightarrow X_0, X_1, X_3, \dots, X_8, X_{10} = 0$



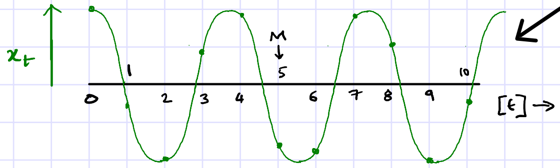
SINUSOID AT 3^{rd} HARMONIC: FREQUENCY $= 3 \times \frac{2\pi}{N}$

TD

→ $x(t) = A_3 \cos\left(3\, \frac{2\pi}{N}\, t + \theta_3\right)$ ← REAL

→ PHASOR: $\frac{A_3}{2} e^{j\theta_3}$, at $\omega = 3\, \frac{2\pi}{N}$

→ $x_t = x(t)$, $t = 0, \ldots, N-1$

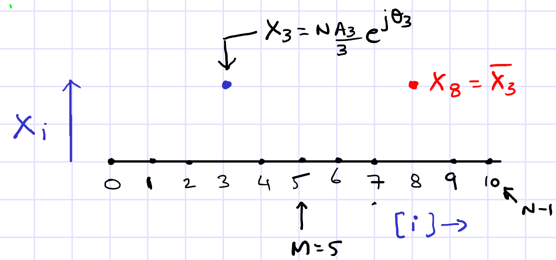


FD

→ $X_3 = N \frac{A_3}{2} e^{j\theta_3}$ ← COMPLEX

→ $X_{N-3} = X_8 = \overline{X_3}$

→ $X_0, X_1, X_2, X_4, X_5, X_6, X_7, X_9, X_{10} = 0$



SINUSOID AT M^{th} HARMONIC: FREQUENCY $= M \times \frac{2\pi}{N}$

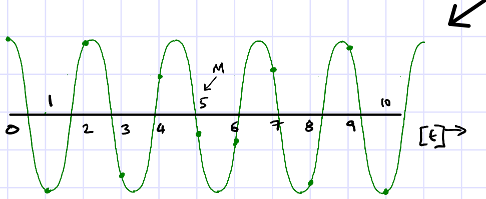
← 5th

TD

→ $x(t) = A_M \cos\left(M\, \frac{2\pi}{N}\, t + \theta_M\right)$ ← REAL

→ PHASOR: $\frac{A_M}{2} e^{j\theta_M}$, at $\omega = \frac{2\pi}{N}$

→ $x_t = x(t)$, $t = 0, \ldots, N-1$

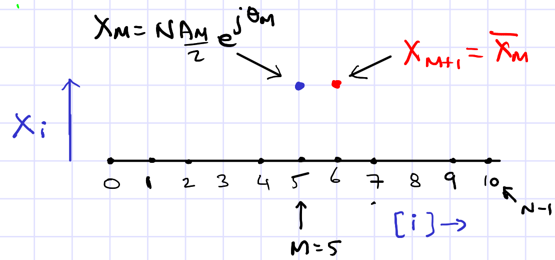


FD

→ $X_M = N \frac{A_M}{2} e^{j\theta_M}$ ← COMPLEX

→ $X_{N-M} = X_{M+1} = \overline{X_M}$

→ $X_0, X_1, X_2, \ldots, X_{M-1}, X_{M+2}, \ldots, X_{N-1} = 0$

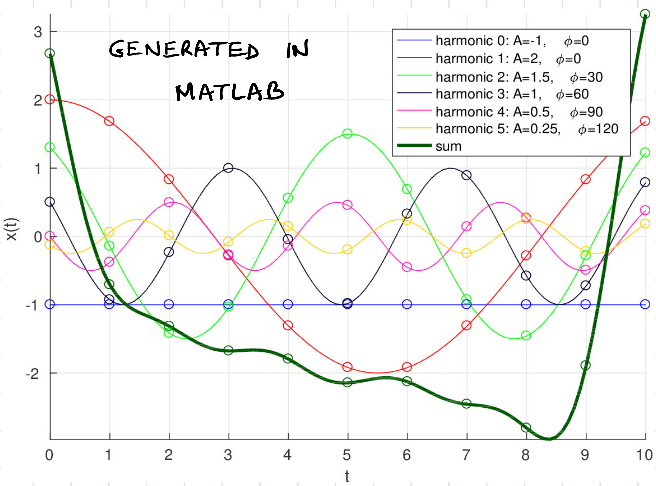


SUM OF ALL THESE SINUSOIDS AT HARMONICALLY RELATED FREQUENCIES (+ DC)

TD

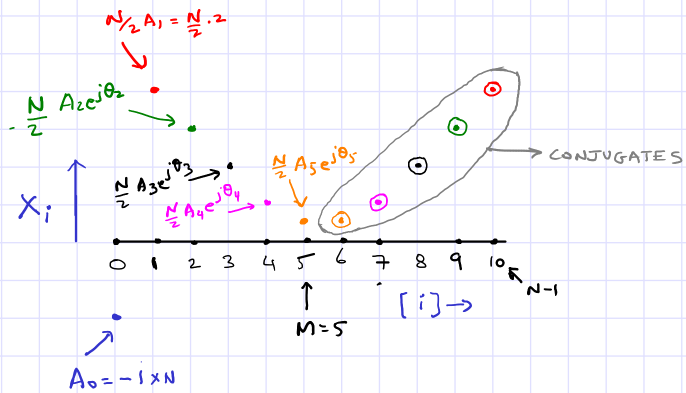
$x(t) = A_0 + \sum_{i=1}^{M} A_i \cos\left(\frac{2\pi}{N} i\, t + \theta_i\right)$ ← REAL

$x_t = x(t)$, $t = 0, \ldots, N-1$



FD

$\vec{X} = F_N\, \vec{x}$



→ KEY POINT

→ TAKE ANY odd $N > 1$, $N = 2M+1$

→ SOMEONE GIVES YOU N SAMPLES OF $x(t) = A_0 + \sum_{i=1}^{M} A_i \cos\left(\frac{2\pi}{N} t + \theta_i\right)$ at $t = 0, 1, \ldots, N-1$

→ WITHOUT TELLING YOU A_0, A_i, θ_i — i.e, YOU DON'T KNOW THE PHASORS $\frac{A_i e^{j\theta_i}}{2}$, or A_0

→ JUST TAKE A DFT : $X = F_N \vec{x}$ — VECTOR OF THE SAMPLES

→ $\frac{1}{N} X_i$ ARE YOUR PHASORS $(i = 1, \ldots, M)$; $\frac{1}{N} X_0$ IS THE DC TERM !

→ ALL YOU NEED ARE THE SAMPLES : THE DFT GETS YOU THE PHASORS IMMEDIATELY

→ CHANGING THE TIME PERIOD / FUNDAMENTAL FREQUENCY

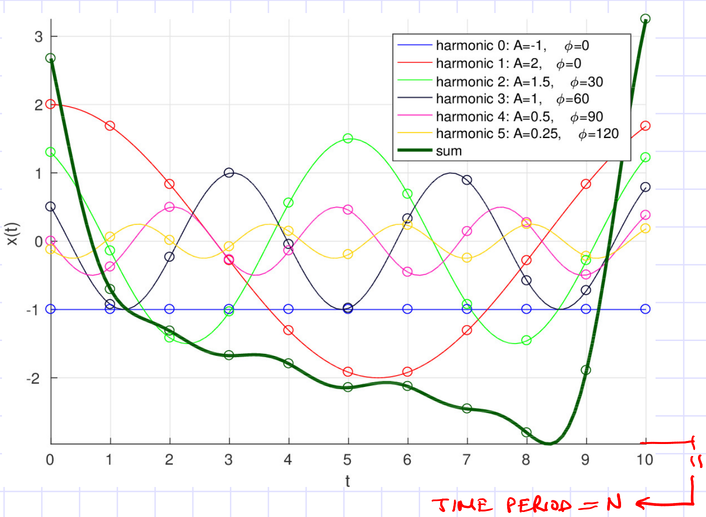
→ we had $x(t) = A_0 + \sum_{i=1}^{M} A_i \cos\left(\frac{2\pi}{N} i\, t + \theta_i\right)$

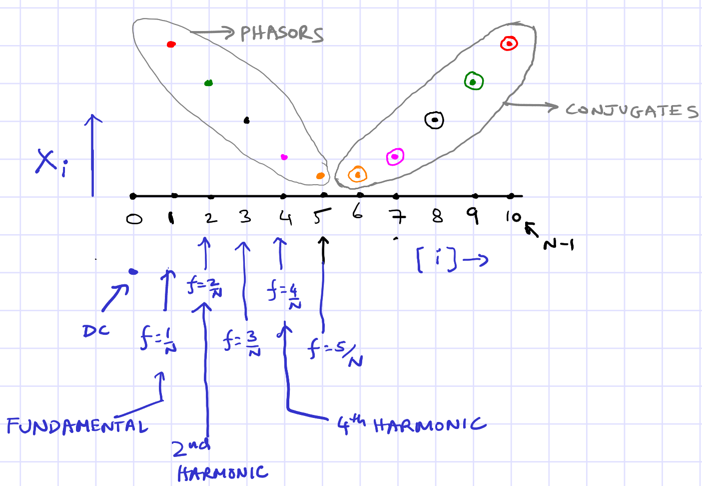
→ THE TIME PERIOD IS N ← RECALL : $\cos\left(\frac{2\pi}{T} t\right)$ has a period of T.

→ FUNDAMENTAL FREQ = $\frac{1}{\text{TIME PERIOD}} = \frac{1}{N}$

frequency of i^{th} harmonic

$$x(t) = A_0 + \sum_{f=1}^{M} A_f \cos\left(2\pi \left(\frac{i}{N}\right) t + \theta_i\right)$$



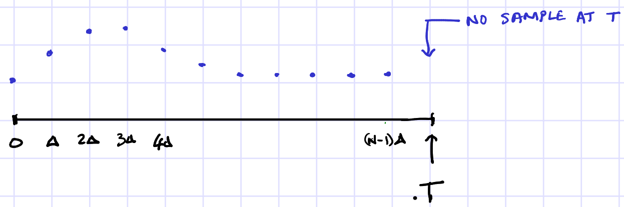


→ SUPPOSE WE WANT A DIFFERENT FUNDAMENTAL FREQUENCY

→ $x(t) = A_0 + \sum_{i=1}^{M} A_f \cos\left(2\pi f_0 i t + \theta_i\right)$

⇒ TIME-PERIOD $T = \frac{1}{f_0}$

→ I STILL TAKE $N = 2M+1$ SAMPLES, SPACED AT $\Delta = \frac{T}{N}$



$$\longrightarrow \quad \vec{x} = \begin{array}{c} \begin{bmatrix} x(0) \\ x(\Delta) \\ x(2\Delta) \\ \vdots \\ x((N-1)\Delta) \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x_0 \\ \leftarrow x_1 \\ \\ \vdots \\ \leftarrow x_{N-1} \end{array}$$

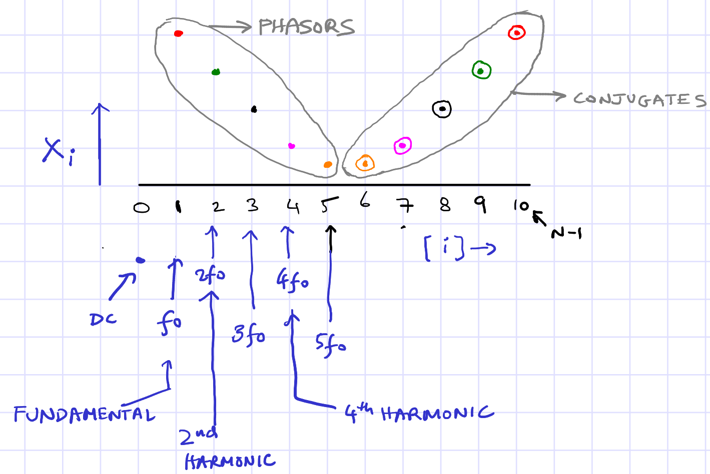
→ IS THE DFT OF $\vec{x}$ STILL USEFUL?

→ YES: IT STILL GIVES YOU THE PHASORS

→ THE FUNDAMENTAL AND HARMONIC FREQUENCIES ARE DIFFERENT

→ $f_0, 2f_0, \ldots, Mf_0$

→ INSTEAD OF $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{M}{N}$



DEMO: CALLING ELVIS IN MATLAB

→ CD FORMAT: SAMPLE RATE = 44,100 SAMPLES PER SECOND

⇒ $\Delta = \frac{1}{44{,}100}$ s ← SPACING OF TD SAMPLES

→ WE'LL TAKE 1 minute = 60 s OF AUDIO DATA

= $60 \times 44{,}100 = 2{,}646{,}000$ SAMPLES

→ WE'LL TAKE $N = 2{,}646{,}001$ (TO MAKE IT ODD)

⇒ $M = \frac{N-1}{2} = 1{,}323{,}000$

→ $T = N \cdot \Delta \Rightarrow T = \dfrac{2,646,001}{44,100} = 60 + 2.267 \times 10^{-5} \simeq 60\,s.$

→ FUNDAMENTAL FREQ $f_0 = \dfrac{1}{T} = 1.667 \times 10^{-2}$ Hz.

→ HIGHEST HARMONIC $= M f_0 = \dfrac{M}{T} = \dfrac{M}{N\Delta} = \dfrac{M}{N} \times 44,100 = \dfrac{M}{2M+1} \cdot 44,100 \simeq 22,050$ Hz

→ DFT SIZE : $N \times N$

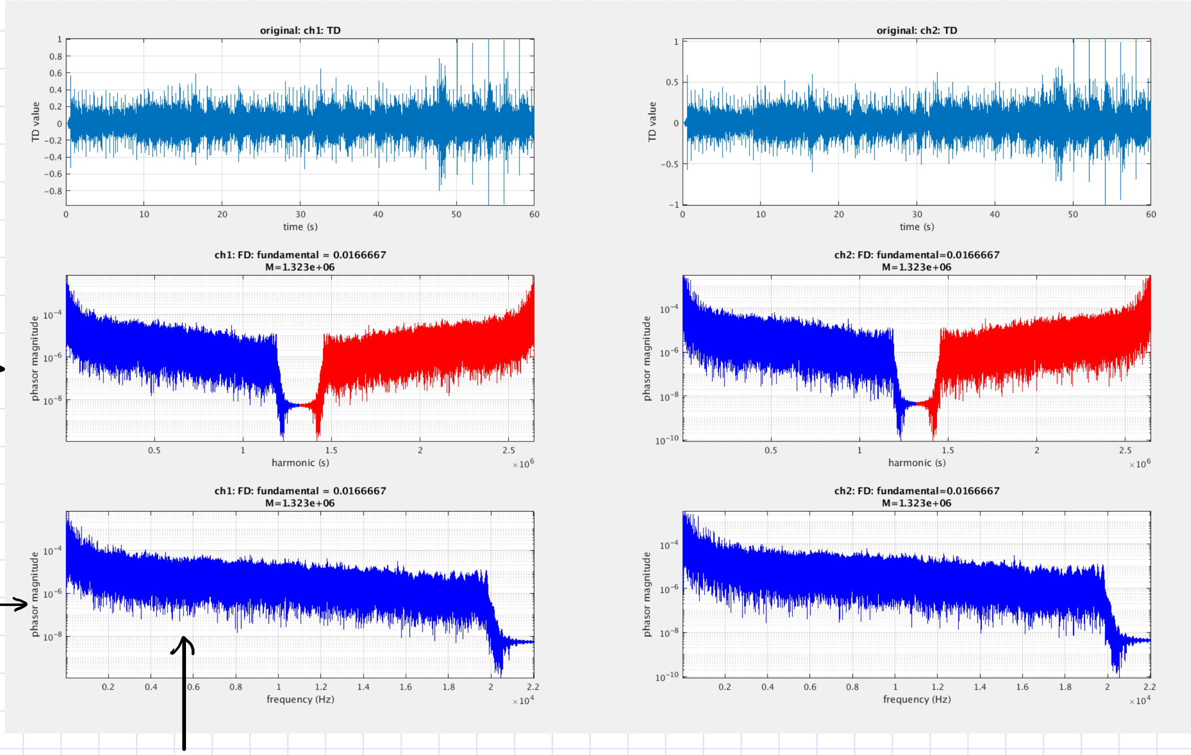
→ $F_N \vec{x}$: HOW MANY MULTIPLICATION OPERATIONS?

→ MY COMPUTER'S CLOCK PERIOD: $\sim \dfrac{1}{4\text{GHz}} = 0.25 \times 10^{-9}$ s ← MINIMUM TIME NEEDED FOR ANY OPERATION

→ COMPUTER TIME NEEDED FOR $F_N \vec{x}$: ? $\times 0.25 \times 10^{-9} =$

→ RUN MATLAB CODE callingelvis.m.

$\dfrac{X_i}{N} = \dfrac{A_i}{2} e^{j\theta_i}$ →



$X(f), f = i f_0$ →
$i = 0, \ldots, M$

FREQUENCY SPECTRUM OF THE AUDIO SIGNAL

→ RECALL (?) THE INVERSE DFT (IDFT)

→ $\vec{X} = F_N \vec{x} \iff \vec{x} = F_N^{-1} \vec{X}$

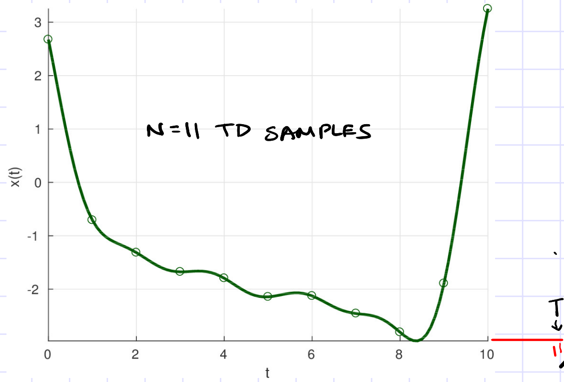
→ CAN SHOW (see notes) that $F_N^{-1} = \dfrac{1}{N} F_N^*$

INTERPOLATION VIA THE DFT

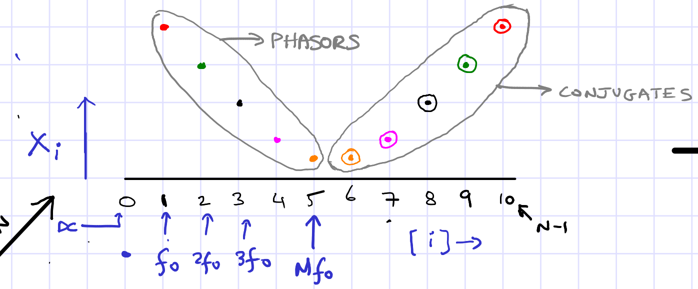
$$x(t) = A_0 + \sum_{i=1}^{M} A_f \cos\left(2\pi f_0 i t + \theta_i\right)$$

TD

FD



F_N



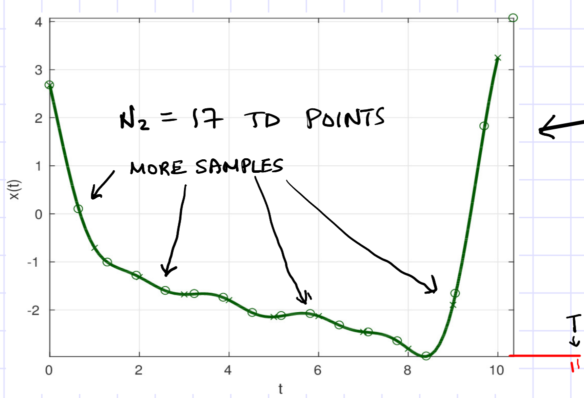
ZERO PAD

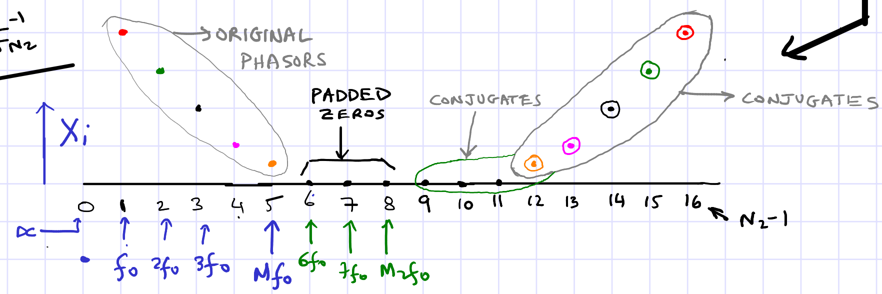
→ INCREASE M IN THE F.D. RERESENTION:

→ $M_2 =$ (say) $8 \Rightarrow N_2 = 2M_2 + 1 = 17$

→ DON'T CHANGE THE SPECTRUM : JUST PAD WITH ZEROS

$F_{N_2}^{-1}$





MORE SAMPLES VIA "BAND LIMITED" INTERPOLATION

→ f_0 (FUNDAMENTAL) REMAINS THE SAME

→ SPECTRUM REMAINS THE SAME

→ $x(t) = A_0 + \sum_{i=1}^{M_2} A_f \cos\left(2\pi f_0 i t + \theta_i\right)$

→ with $A_6 = A_7 = A_8 = 0$: NO CHANGE TO $x(t)$

→ DFT-BASED (BAND-LIMITED) INTERPOLATION : THE PROCEDURE

→ START with $\vec{x} \in \mathbb{R}^N$, $N = 2M+1$ ← representing samples at $0, \frac{T}{N}, \frac{2T}{N}, \dots, \frac{(N-1)T}{N}$

→ $\vec{X} = F_N \vec{x}$ (SIZE N DFT)

→ CHOOSE SOME $M_2 > M$; DEFINE $N_2 = 2M_2 + 1$

→ DEFINE $\vec{X}_2$ BY :

→ $\vec{X}_2[0,\dots,M] = \frac{N_2}{N} \vec{X}[0,\dots,M]$ ← COPY THE PHASORS OF THE ORIGINAL HARMONICS

→ $\vec{X}_2[N_2-M,\dots,N_2-1] = \frac{N_2}{N} \vec{X}[M+1,\dots,N-1]$ ← COPY THE CONJUGATES TO THE RIGHT PLACES

→ $\vec{X}_2[M+1,\dots,N_2-M-1] = 0$ ← PAD THE EXTRA HARMONICS TO ZEROS.

→ $\vec{x}_2 = F_{N_2}^{-1} \vec{X}_2$ (SIZE N_2 INVERSE DFT)

INTERPOLATED SAMPLES at $0, \frac{T}{N_2}, \frac{2T}{N_2}, \dots, \frac{(N_2-1)T}{N_2}$