

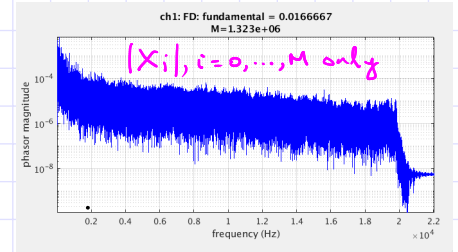
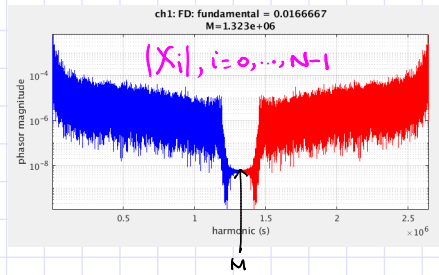
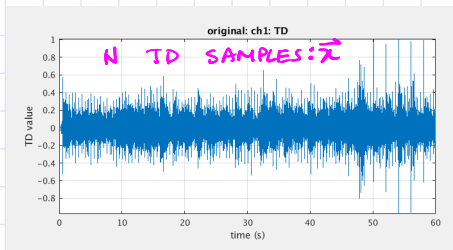
THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

1. INTRODUCTION AND PRELIMINARIES ✓
 2. ROOTS OF 1 ON THE COMPLEX PLANE ✓
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8. FILTERING VIA THE DFT

→ RECALL: (60+ε)s OF CALLING-ELVIS: $T \approx 60s$, $f_0 = \frac{1}{T} \approx \frac{1}{60} \text{ Hz}$, $M = 1,323,000$, $N = 2M + 1 = 2,626,001$

$$\vec{X} = F_N \vec{x}$$



→ RECALL: FREQ. DOMAIN INTERPRETATION:

→ $\vec{x}$ are samples of $x(t) = \underbrace{\frac{X_0}{N}}_{\text{DC}} + \sum_{i=1}^M \underbrace{\left(\frac{2}{N} |X_i| \right)}_{\text{magnitude of } X_i} \cos(2\pi f_0 i t + \underbrace{\angle X_i}_{\text{angle of } X_i = 0_i})$

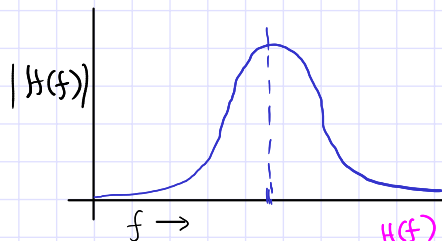
$\sum_{i=1}^M X_i$: PHASOR at freq = f_0

$$X_i = |X_i| e^{j\theta_i}$$

→ FILTERING:

→ MODIFY PHASOR AT any freq. f by $\underbrace{H(f)}_{\text{TRANSFER FUNCTION OF FILTER}}$ (MULTIPLYING WITH)

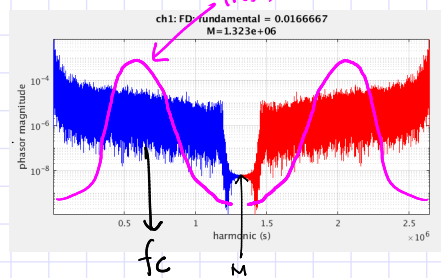
→ EXAMPLE: BAND-PASS FILTER



→ Eg: $H(f) = \frac{1}{\left(\frac{f-f_c}{\Delta B}\right)^2 + 1}$

↑ SIMILAR TO $H(f)$ OF AN RLC CIRCUIT

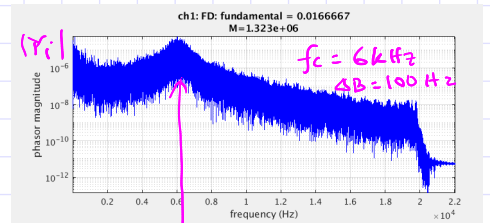
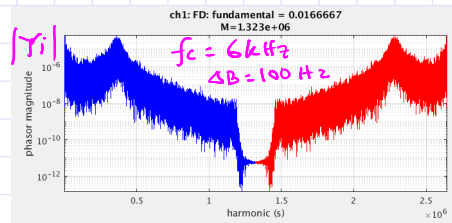
→ APPLY →



$$Y_i \mapsto X_i H(if_0), \quad i=0, \dots, M$$

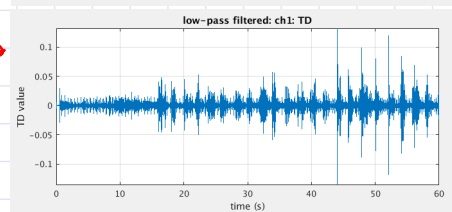
$$Y_{N-i} = \overline{X_i}, \quad i=0, \dots, M \leftarrow \text{COMPLEX CONJUGATES}$$

→ FILTERED: FD.



DEMO: PLAY IN MATLAB →

TD



9. CONVOLUTION AND THE DFT

9A: → RECALL CONVOLUTION:

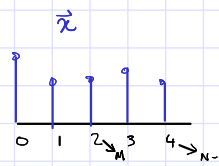
$$\rightarrow u[t] \otimes h[t] \triangleq \sum_{i=-\infty}^{\infty} u[i] h[t-i]$$

$$= \sum_{i=-\infty}^t u[i] h[t-i] \quad (\text{IF } h[t] \text{ CAUSAL})$$

$$= \sum_{i=0}^t u[i] h[t-i] \quad (\text{IF } h[t] \text{ and } u[t] \text{ BOTH CAUSAL})$$

9B: CIRCULAR CONVOLUTION

→ GIVEN $\vec{x}$ & $\vec{y}$, both $\in \mathbb{R}^N$



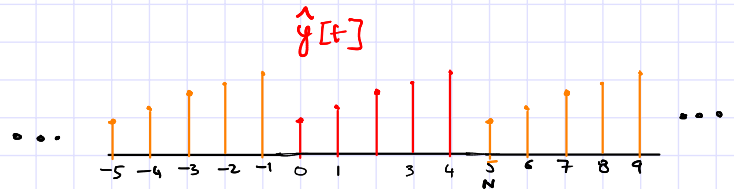
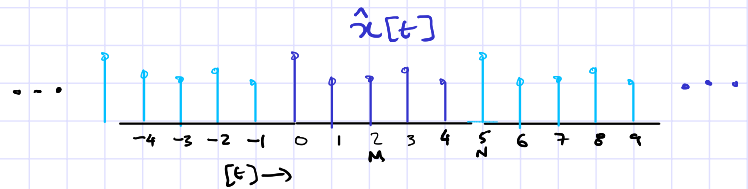
→ EXTEND BOTH PERIODICALLY (CIRCULARLY)

$$\rightarrow \hat{x}[t] = x[t], \quad t=0, \dots, N-1$$

$$\rightarrow \hat{x}[t] = \hat{x}[t+kN], \quad \forall t, \forall k \in \mathbb{Z}$$

$$\rightarrow \hat{y}[t] \triangleq y[t], \quad t=0, \dots, N-1$$

$$\rightarrow \hat{y}[t] = \hat{y}[t+kN] \quad \forall t, \forall k \in \mathbb{Z}$$



→ NOW DEFINE CIRCULAR

CONVOLUTION TO BE:

$$\hat{z}[t] \triangleq \sum_{i=0}^{N-1} \hat{x}[i] \hat{y}[t-i], \quad \forall t.$$

→ WE'LL WRITE IT AS:

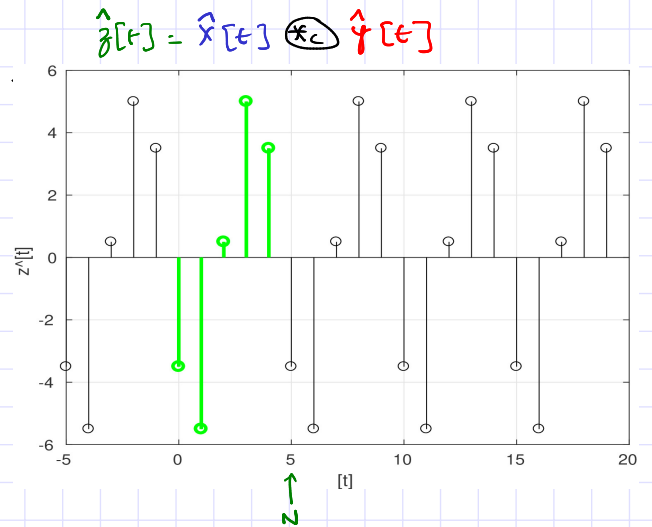
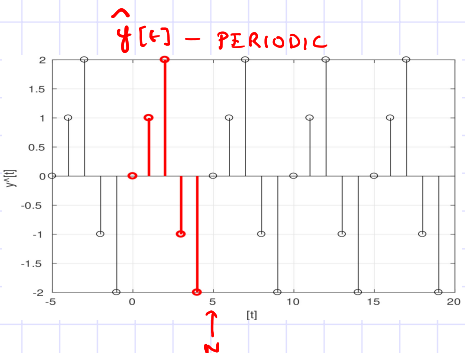
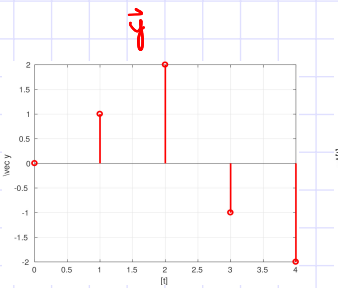
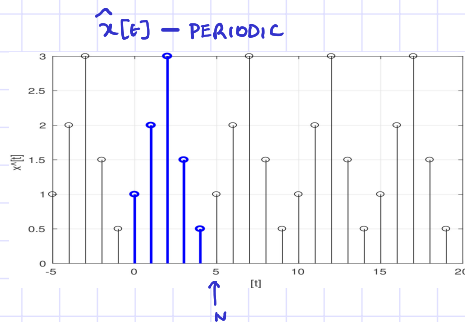
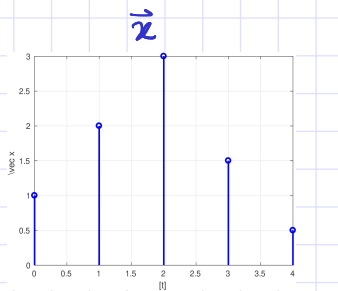
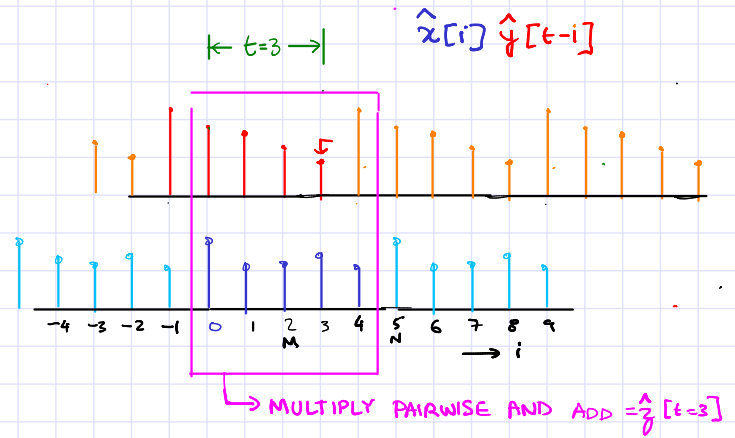
$$\hat{z}[t] = \hat{x}[t] \circledast \hat{y}[t]$$

↑
CIRCULAR CONVOLUTION

→ CLAIM: $\hat{z}[t]$ is periodic/circular

→ Proof:

$$\begin{aligned} \hat{z}[t+kN] &= \sum_{i=0}^{N-1} \hat{x}[i] \hat{y}[t-i+kN] \\ &= \sum_{i=0}^{N-1} \hat{x}[i] \hat{y}[t-i] = \hat{z}[t] \end{aligned}$$



9C: DFT and CIRCULAR CONVOLUTION

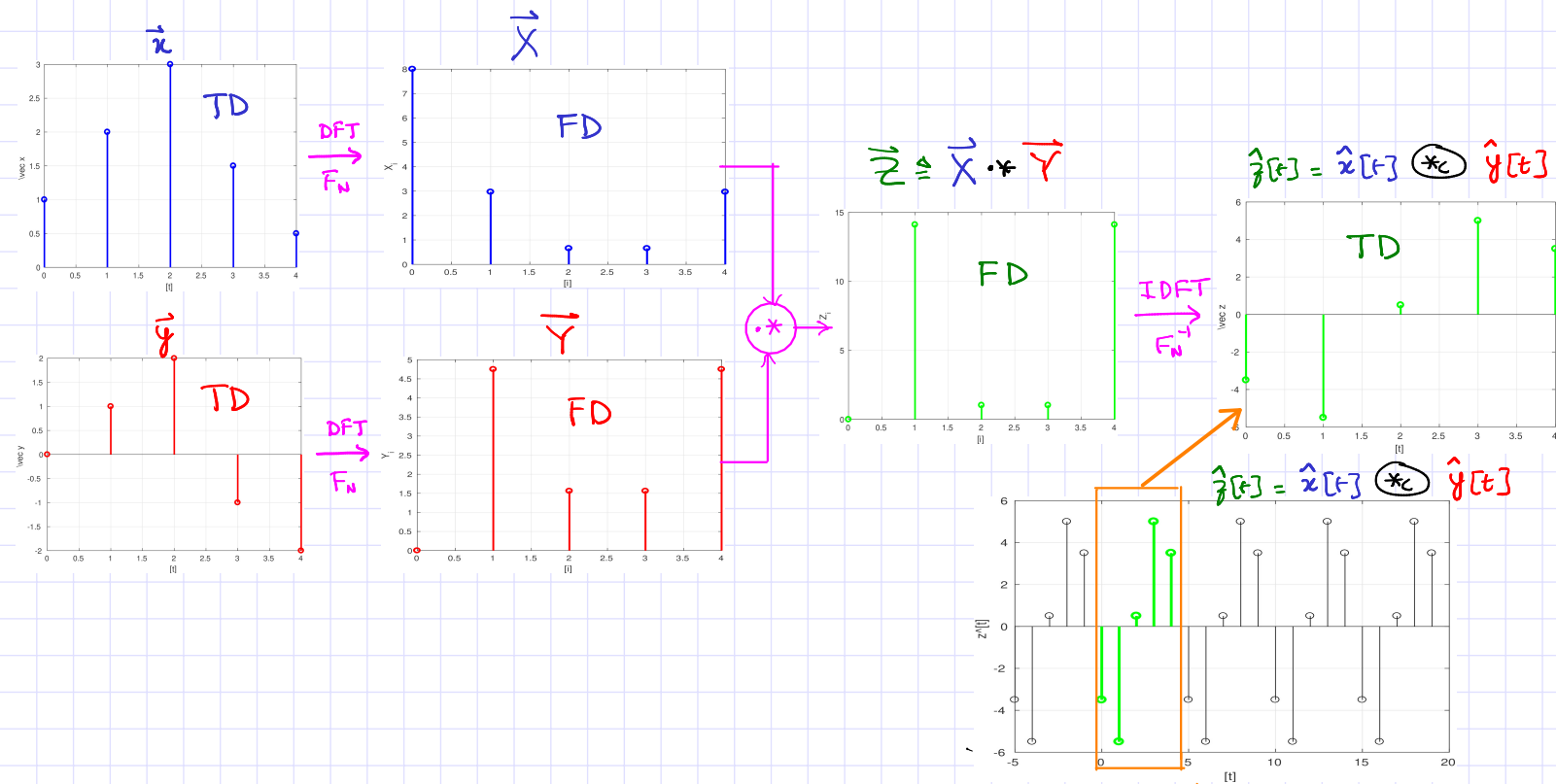
→ KEY PROPERTY OF DFTs:

$$\left[\begin{array}{l} \rightarrow \text{EXTEND } \vec{x} \text{ and } \vec{y} \text{ periodically to } \hat{x}[k] \text{ \& } \hat{y}[k], \text{ resp.} \\ \rightarrow \text{Define } \hat{z}[k] = \hat{x}[k] \circledast \hat{y}[k] \quad \leftarrow \text{CIRCULAR CONVOLUTION} \\ \rightarrow \text{Define } \vec{z} = [\hat{z}[0], \hat{z}[1], \dots, \hat{z}[N-1]]^T \end{array} \right]$$

→ LET $\vec{X} = F_N \vec{x}$, $\vec{Y} = F_N \vec{y}$, $\vec{Z} = F_N \vec{z}$ (i.e., take the DFTs of $\vec{x}$, $\vec{y}$, and $\vec{z}$)

→ then: $\vec{Z} = \vec{X} \cdot \vec{Y} =$
 $\begin{bmatrix} x_0 y_0 \\ x_1 y_1 \\ \vdots \\ x_{N-1} y_{N-1} \end{bmatrix}$

ELEMENT BY ELEMENT MULTIPLICATION



→ DFT TURNS CIRCULAR CONVOLUTION IN THE TIME-DOMAIN INTO MULTIPLICATION IN THE FREQUENCY DOMAIN

THE DFT AND LTI SYSTEMS

→ RECALL: LTI SYSTEMS CHARACTERIZED BY IMPULSE RESPONSE $h[t]$

$$\rightarrow y[t] = u[t] \otimes h[t] = \sum_{i=0}^t u[i] h[t-i] \quad (\text{assuming } h[t] \text{ \& } u[t] \text{ both causal})$$

↑ NOT CIRCULAR CONV.

→ Q: SUPPOSE I WANT TO CALCULATE $y[0], \dots, y[n]$

→ HOW MANY MULTIPLICATION OPERATIONS DO I NEED?

→ FOR $y[0]: 1; y[1]: 2; y[2]: 3, \dots, y[n]: n+1$

$$\Rightarrow 1+2+\dots+(n+1) = \frac{(n+1)(n+2)}{2} \sim \frac{n^2}{2} \text{ if } n \text{ is large.}$$

→ TRY $n \sim 2 \times 10^6$ (1 minute of CD AUDIO) $\Rightarrow \frac{n^2}{2} \sim 2 \times 10^{12} = 2 \text{ TRILLION OPS}$
↳ ~10m COMPUTER TIME

→ CAN WE DO IT (MUCH) FASTER?

→ YES: USING THE DFT AND CIRCULAR CONVOLUTION

→ STRATEGY

→ FIRST, FIX THE NUMBER OF $y[t]$ VALUES YOU WANT: $y[0], \dots, y[N-1]$ DECIDE N

→ A: EXPRESS $y[t]$ AS A CIRCULAR CONVOLUTION, NOT JUST A REGULAR CONVOLUTION

$$\rightarrow y[t] = u[t] \otimes h[t]$$

$$\stackrel{?}{=} \hat{u}_p[t] \otimes_c \hat{h}_p[t], \quad t = 0, \dots, N-1$$

→ CAN WE FIND SUITABLE $\hat{u}[t]$ AND $\hat{h}_p[t]$?

→ B: IF WE CAN, THEN THE DFT CAN PERFORM CIRCULAR CONVOLUTION:

$$\rightarrow \vec{\hat{y}}_p \triangleq (\text{DFT of } \vec{\hat{u}}_p * (\text{DFT of } \vec{\hat{h}}_p))$$

$$\rightarrow \vec{\hat{y}}_p = \text{IDFT of } \vec{\hat{y}}_p$$

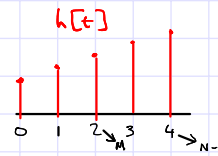
→ THE FIRST N ENTRIES SHOULD BE $y[0], \dots, y[N-1]$

9D: EXPRESSING CONVOLUTION AS CIRCULAR CONVOLUTION

→ WE HAVE: $y[t] = \sum_{i=0}^t u[i] h[t-i]$ (u, h, BOTH CAUSAL)

→ WE ARE INTERESTED ONLY IN $t = 0, \dots, N-1$

→ WILL NEED ONLY $u[0], \dots, u[N-1]$ AND $h[0], \dots, h[N-1]$



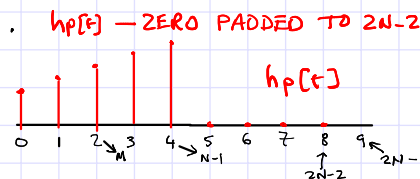
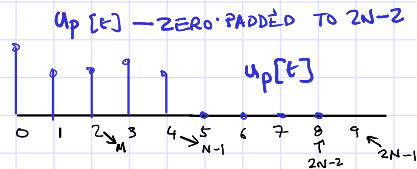
→ DEFINE $u_p[t]$ and $h_p[t]$: EXTENDED AND ZERO PADDED VERSIONS OF $u[t]$ / $h[t]$

→ $u_p[t] = u[t]$, $t = 0, \dots, N-1$ (N original TD values)

→ $u_p[N], u_p[N+1], \dots, u_p[2N-2] = 0$ ← (ADDITIONAL N-1 ZEROS)

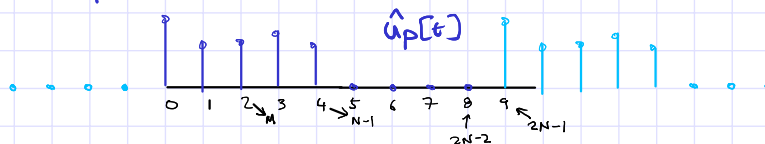
→ $h_p[t] = h[t]$; $t = 0, \dots, N-1$ (N original TD values)

→ $h_p[N], h_p[N+1], \dots, h_p[2N-2] = 0$ ← (ADDITIONAL N-1 ZEROS)



→ NOW EXTEND BOTH PERIODICALLY (MAKE THEM CIRCULAR)

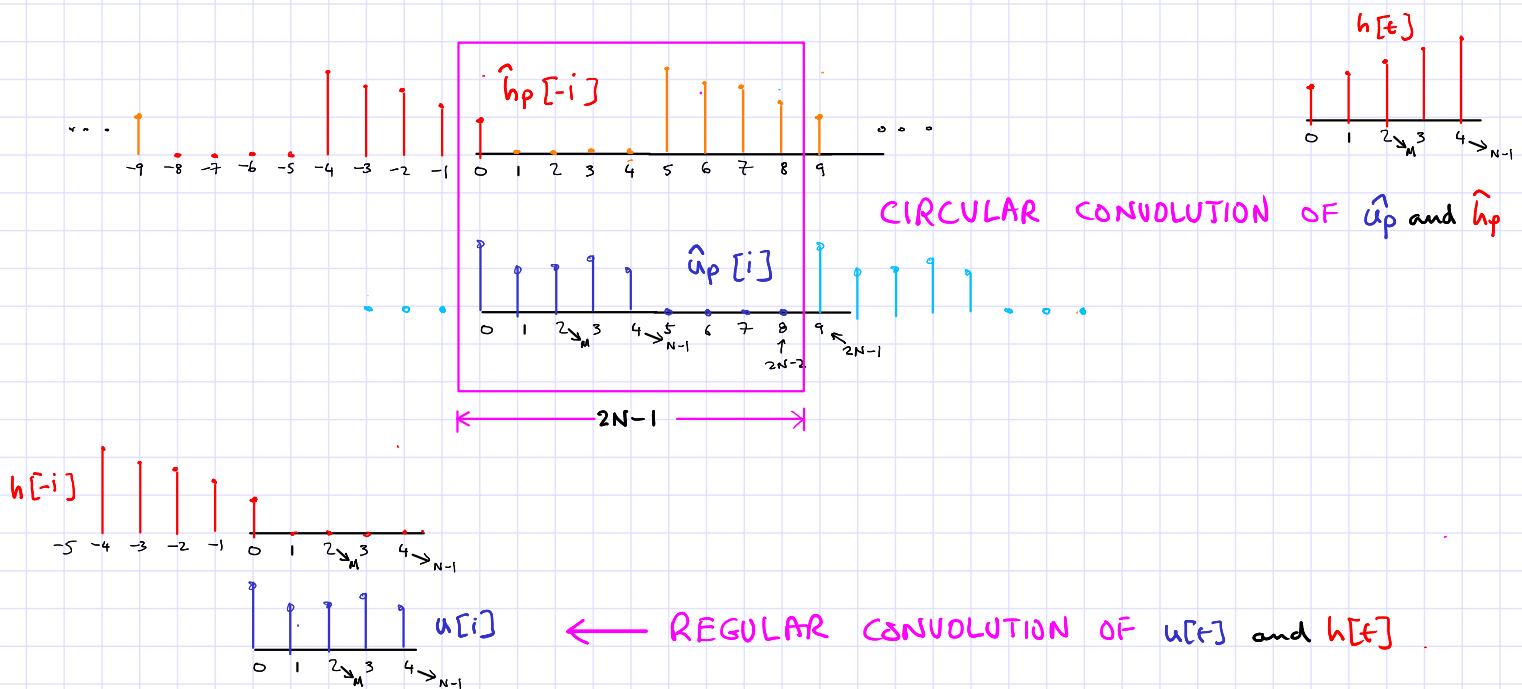
$\hat{u}_p$: u_p ZERO PADDED TO $2N-2$ AND REPEATED (WITH PERIOD $2N-1$)



$\hat{h}_p$: h_p ZERO PADDED TO $2N-2$ AND REPEATED (W PERIOD $2N-1$)



→ WHAT DOES THE CIRCULAR CONVOLUTION OF $\hat{u}_p$ AND $\hat{h}_p$ LOOK LIKE?
 → VS REGULAR CONVOLUTION?



→ IDENTICAL FOR $t = 0, \dots, N-1$!

→ HENCE, CAN USE DFTs FOR CONVOLUTION

9E. STRATEGY FOR FAST CONVOLUTION USING DFTs

→ CHOOSE $N = 2M+1$ (odd)

→ set $\vec{u}_p = \begin{cases} u[t], & t=0, \dots, N-1 \\ 0, & t=N, \dots, 2N-2 \end{cases} \vec{u}_p \in \mathbb{R}^{2N-1}$

→ set $\vec{h}_p = \begin{cases} h[t], & t=0, \dots, N-1 \\ 0, & t=N, \dots, 2N-2 \end{cases} \vec{h}_p \in \mathbb{R}^{2N-1}$

→ RUN 2 DFTs: $\vec{U}_p \triangleq F_{2N-1} \vec{x}_p$, $\vec{H}_p \triangleq F_{2N-1} \vec{x}_p$

→ SET $\vec{Y}_p = \vec{X}_p \circledast \vec{H}_p$
 ↳ ELEM. BY ELEM. MULT.

→ RUN an INVERSE DFT: $\vec{y}_p = F_{2N-1}^{-1} \vec{Y}_p$

→ $y[0] = y_p[0] \dots, y[N-1] = y_p[N-1] (= u[t] \circledast h[t])$

→ IGNORE $y_p[N], \dots, y_p[2N-2]$

COMPUTATION

FFT: $\sim 2N \log(2N) \times 2$

$\circledast$: $\sim 2N$ MULTIPLICATIONS

FFT: $\sim 2N \log(2N)$

→ $6N \log(2N) + 2N$

→ vs. $\frac{N^2}{2}$

→ TRY $N = 2 \times 10^6$

→ FFT BASED: 2.67×10^8

→ DIRECT: 2×10^{12}

→ APPENDIX:

— DFT TURNS CIRCULAR CONVOLUTION INTO MULTIPLICATION: PROOF

$$\rightarrow \text{GIVEN } z[t] = x[t] \circledast y[t] = \sum_{i=0}^{N-1} x[i] y[t-i]$$

$$\begin{aligned} \rightarrow z_i &= \sum_{j=0}^{N-1} \omega_N^{-ij} z[j] = \sum_{j=0}^{N-1} \omega_N^{-ij} \sum_{k=0}^{N-1} x[k] y[j-k] = \sum_{k=0}^{N-1} x[k] \sum_{j=0}^{N-1} \omega_N^{-ij} y[j-k] \\ &= \sum_{k=0}^{N-1} x[k] \left[\underbrace{\sum_{j=0}^{k-1} \omega_N^{-ij} y[j-k]}_{\text{CALL THIS } A_k} + \underbrace{\sum_{j=k}^{N-1} \omega_N^{-ij} y[j-k]}_{\text{CALL THIS } B_k} \right] = \sum_{k=0}^{N-1} x_k (A_k + B_k) \end{aligned}$$

$$\rightarrow A_k = \sum_{j=0}^{k-1} \omega_N^{-ij} y[j-k]$$

$$\rightarrow \text{From circularity: } y[j-k] = y[j-k+N]$$

$$\Rightarrow A_k = \sum_{j=0}^{k-1} \omega_N^{-ij} y[j-k] = \sum_{j=0}^{k-1} \omega_N^{-ij} y[j-k+N]$$

$$\rightarrow \text{Define } l = j-k+N \Rightarrow j = l+k-N$$

$$\Rightarrow A_k = \sum_{j=0}^{k-1} \omega_N^{-ij} y[j-k+N] = \sum_{l=N-k}^{N-1} \omega_N^{-i(l+k-N)} y[l] = \omega_N^{-ik} \sum_{l=N-k}^{N-1} \omega_N^{-il} y[l]$$

$$\rightarrow B_k = \sum_{j=k}^{N-1} \omega_N^{-ij} y[j-k]$$

$$\rightarrow \text{Let } l = j-k \Rightarrow j = l+k$$

$$\Rightarrow B_k = \sum_{j=k}^{N-1} \omega_N^{-ij} y[j-k] = \sum_{l=0}^{N-k-1} \omega_N^{-i(l+k)} y[l] = \omega_N^{-ik} \sum_{l=0}^{N-k-1} \omega_N^{-il} y[l]$$

$$\Rightarrow A_k + B_k = \omega_N^{-ik} \sum_{l=N-k}^{N-1} \omega_N^{-il} y[l] + \omega_N^{-ik} \sum_{l=0}^{N-k-1} \omega_N^{-il} y[l] = \omega_N^{-ik} \underbrace{\sum_{l=0}^{N-1} \omega_N^{-il} y[l]}_{Y_i} = \omega_N^{-ik} Y_i$$

$$\Rightarrow z_i = \sum_{k=0}^{N-1} (A_k + B_k) x_k = \sum_{k=0}^{N-1} \omega_N^{-ik} x_k Y_i = \underbrace{Y_i \sum_{k=0}^{N-1} \omega_N^{-ik} x_k}_{X_i} = X_i Y_i \leftarrow \text{PROVED.}$$

→ ALGEBRAIC PROOF THAT $\hat{u}[t] \circledast \hat{h}[t] = u[t] \circledast h[t]$ for $t=0, \dots, N-1$

$$\rightarrow \hat{u}[t] \circledast \hat{h}[t] = \sum_{i=0}^{2N-2} \hat{u}[i] \hat{h}[t-i]$$

$$\Rightarrow \hat{u}[t] \circledast \hat{h}[t] = \sum_{i=0}^{N-1} \hat{u}[i] \hat{h}[t-i] + \sum_{i=N}^{2N-2} \hat{u}[i] \hat{h}[t-i]$$

$0 \rightarrow$ by padding

$$= \sum_{i=0}^{N-1} \hat{u}[i] \hat{h}[t-i]$$

→ WE ARE INTERESTED ONLY IN $t=0, \dots, N-1$

NOTE: $t-i$ ranges from $(-N+1)$ to $(N-1)$

NOTE2: $\hat{h}[-N+1], \hat{h}[-N+2], \dots, \hat{h}[-1] = 0$

→ BECAUSE OF CIRCULARITY and ZERO PADDING:

$$\rightarrow \text{eg: } \hat{h}[-N+1] = \hat{h}[-N+1+2N-1] = \hat{h}[N] = 0$$

CIRCULARITY FIRST ZERO PADDING LOCATION

$$\rightarrow \text{eg: } \hat{h}[-1] = \hat{h}[-1+2N-1] = \hat{h}[2N-2] = 0$$

LAST ZERO PADDING LOCATION

$$\Rightarrow \hat{u}[t] \circledast \hat{h}[t] = \sum_{i=0}^{N-1} \hat{u}[i] \hat{h}[t-i] = \sum_{i=0}^t \hat{u}[i] \hat{h}[t-i], \quad t=0, \dots, N-1$$

$\downarrow$
0 if $i > t$

→ t ranges from 0 to $N-1$

→ i ranges from 0 to $t \Rightarrow t-i$ ranges from 0 to t

→ i and $t-i$ both range from 0 to $N-1$

$$\Rightarrow \hat{u}[i] = u[i] \text{ and } \hat{h}[t-i] = h[t-i]$$

$$\Rightarrow \hat{u}[t] \circledast \hat{h}[t] = \sum_{i=0}^t u[i] h[t-i] = u[t] \circledast h[t], \quad t=0, \dots, N-1$$

→ SUMMARY:

FROM $t=N$ to $t=2N-2$

WITH PERIOD $2N-1$

→ ZERO-PAD $u[t]$ and $h[t]$, THEN MAKE CIRCULAR/PERIODIC, CALL THESE $\hat{u}[t]$ and $\hat{h}[t]$

$$\Rightarrow \text{then } \hat{u}[t] \circledast \hat{h}[t] = u[t] \circledast h[t] \text{ for } t=0, \dots, N$$