THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

1. INTRODUCTION AND PRELIMINARIES ✓
2. ROOTS OF 1 ON THE COMPLEX PLANE ✓
3. THE DFT MATRIX ✓
4. VECTORS FROM SEQUENCES ✓
5. FREQUENCY INTERPRETATION OF DFT SIGNALS — WITHOUT PROOFS ✓
6. DEMO: MATLAB ✓
7. INTERPOLATION VIA THE DFT ✓
8. FILTERING VIA THE DFT — w MATLAB DEMO ←
9. CONVOLUTION VIA THE DFT, INCLUDING APPLICATION TO LTI SYSTEMS ←

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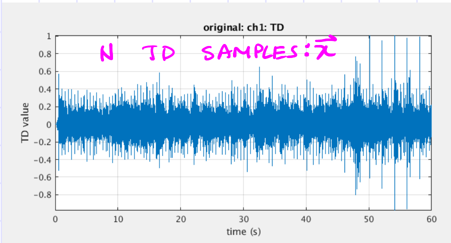
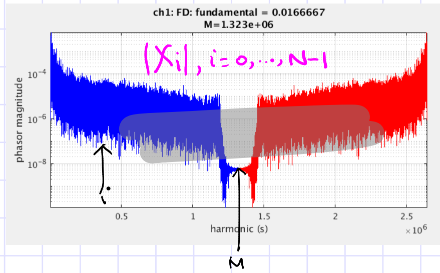
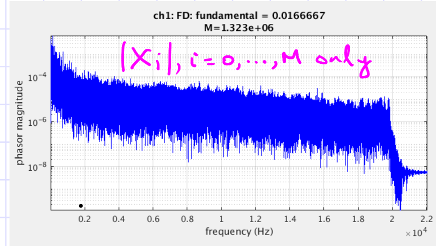
8. FILTERING VIA THE DFT

→ RECALL: $(60+\epsilon)$ s OF CALLING-ELVIS: $T \simeq 60$s, $f_0 = \frac{1}{T} \simeq \frac{1}{60}$ Hz, $M = 1{,}323{,}000$, $N = 2M+1 = 2{,}626{,}001$

$$\vec{X} = F_N \vec{x}$$







$$\boxed{X_i = |X_i| e^{j\theta_i}}$$

→ RECALL: FREQ. DOMAIN INTERPRETATION:

→ $\vec{x}$ are samples of

magnitude of X_i ; angle of $X_i = \theta_i$

$$x(t) = \frac{X_0}{N} + \sum_{i=1}^{M} \frac{2}{N}|X_i| \cos(2\pi f_0 i t + \angle X_i)$$

DC

$A_i \cos(2\pi \underline{f_0 i}\, t + \theta_i)$

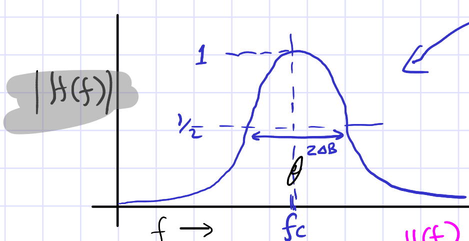
$$A_i e^{j\theta_i} = \frac{2}{N} X_i \quad i=1,\dots,M$$

$\frac{2}{N} X_i$: PHASOR at freq $= i f_0$

→ FILTERING:

→ MODIFY PHASOR AT any freq. f by MULTIPLYING WITH $H(f)$ ← TRANSFER FUNCTION OF FILTER

→ EXAMPLE: BAND-PASS FILTER



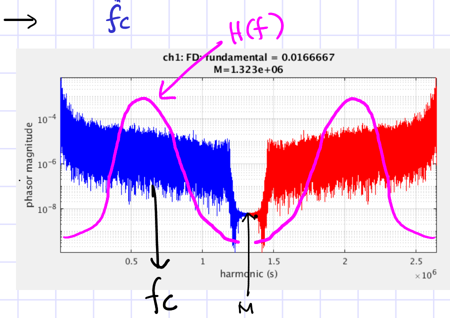
→ Eg:

$$H(f) = \frac{1}{\left(\frac{f-f_c}{\Delta B}\right)^2 + 1}$$

$f = f_c \pm \Delta B \implies |H(f)| = \frac{1}{2}$

SIMILAR TO $H(f)$ OF AN RLC CIRCUIT

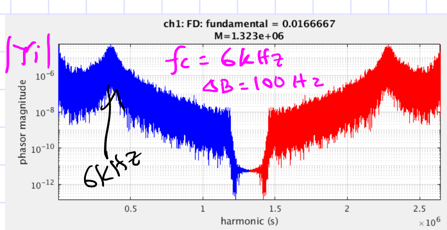
→ APPLY →

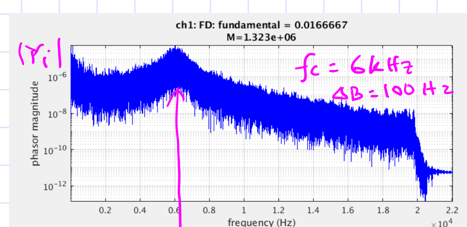


→ $Y_i = X_i H(i f_0)$, $i=0,\dots,M$

→ $Y_{N-i} = \overline{Y_i}$, $i=0,\dots,M$ ← COMPLEX CONJUGATES

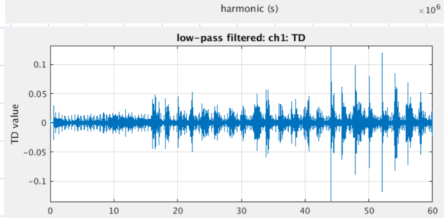
→ FILTERED: FD.

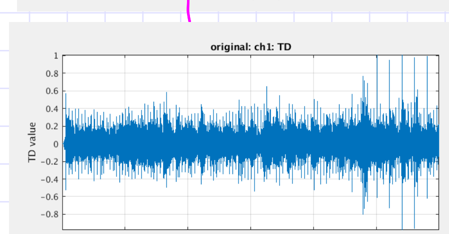




DEMO: PLAY IN MATLAB →

TD





9. CONVOLUTION AND THE DFT

9A:

→ RECALL CONVOLUTION:

$Y[t]$: CAUSAL $\Rightarrow Y[t]=0 \quad \forall t<0$

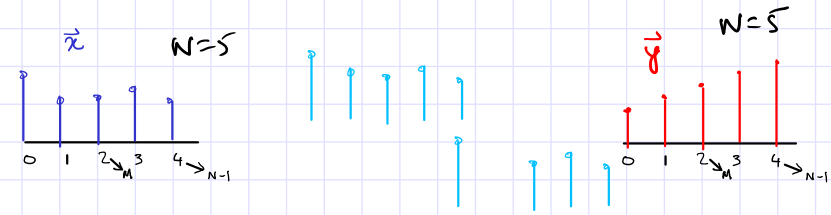
$$\rightarrow z[t] \triangleq u[t] \circledast h[t] \triangleq \sum_{i=-\infty}^{\infty} u[i]\,h[t-i]$$

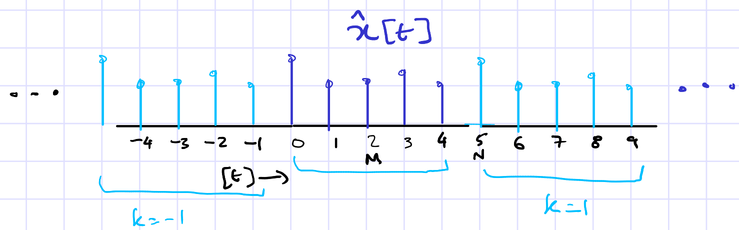
$$= \sum_{i=-\infty}^{t} u[i]\,h[t-i] \quad \text{(IF } h[t] \text{ CAUSAL)}$$

$$= \sum_{i=0}^{t} u[i]\,h[t-i] \quad \text{(IF } h[t] \text{ and } u[t] \text{ BOTH CAUSAL)}$$

9B: CIRCULAR CONVOLUTION

→ GIVEN $\vec{x}$ & $\vec{y}$, both $\in \mathbb{R}^N$





→ EXTEND BOTH PERIODICALLY (CIRCULARLY)

→ $\hat{x}[t] = x[t], \quad t=0,\dots,N-1$

→ $\hat{x}[t] = \hat{x}[t+kN] \quad \forall t, \forall k \in \mathbb{Z}$

→ $\hat{y}[t] \triangleq y[t], \quad t=0,\dots,N-1$

→ $\hat{y}[t] = \hat{y}[t+kN] \quad \forall t \quad \forall k \in \mathbb{Z}$

REGULAR
$\rightarrow \quad z[t] = x[t] \circledast y[t] = \sum_{i=0}^{t} x[i]\, y[t-i]$

$\rightarrow$ NOW DEFINE CIRCULAR

CONVOLUTION TO BE:

CIRCULAR
$$\rightarrow \boxed{\hat{z}[t] \triangleq \sum_{i=0}^{N-1} \hat{x}[i]\, \hat{y}[t-i], \quad \forall\, t.}$$

$\rightarrow$ WE'LL WRITE IT AS:

$$\hat{z}[t] = \hat{x}[t] \circledast_c \hat{y}[t]$$

CIRCULAR CONVOLUTION

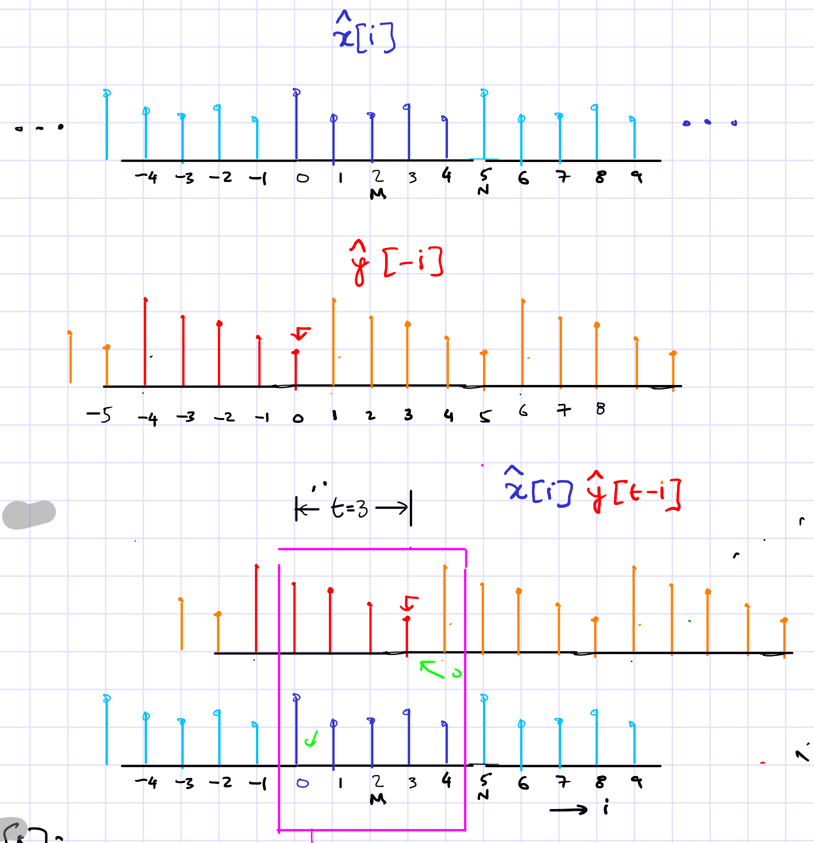
$\rightarrow$ CLAIM: $\hat{z}[t]$ is periodic/circular

$\rightarrow$ Proof:

$$\hat{z}[t+kN] = \sum_{i=0}^{N-1} \hat{x}[i]\, \hat{y}[t-i+kN]$$

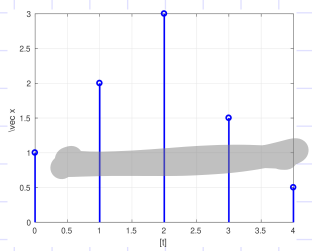
$$= \sum_{i=0}^{N-1} \hat{x}[i]\, \hat{y}[t-i] = \hat{z}[t]$$

$\hat{z}[6] = \hat{z}[1]$

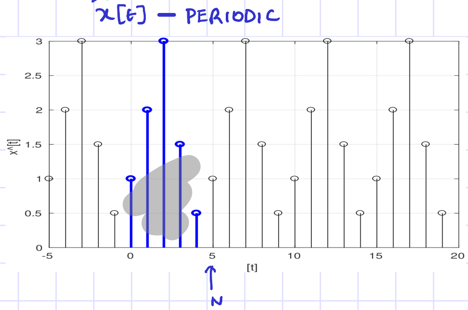


$\hookrightarrow$ MULTIPLY PAIRWISE AND ADD $= \hat{z}[t=3]$

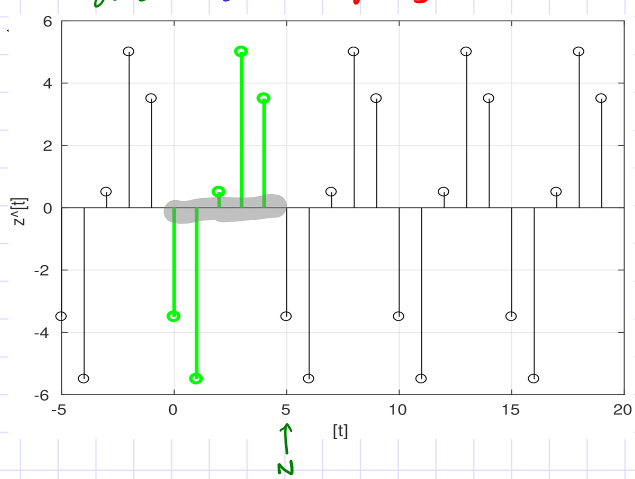
$\vec{x}$



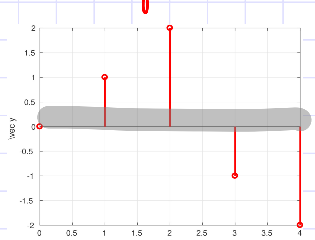
$\hat{x}[t]$ — PERIODIC



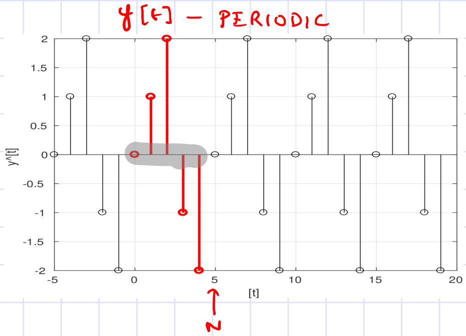
$\hat{z}[t] = \hat{x}[t] \circledast_c \hat{y}[t]$



$\vec{y}$



$\hat{y}[t]$ — PERIODIC



9C: DFT and CIRCULAR CONVOLUTION

→ KEY PROPERTY OF DFTs:

→ EXTEND $\vec{x}$ and $\vec{y}$ periodically to $\hat{x}[t]$ & $\hat{y}[t]$, resp.)

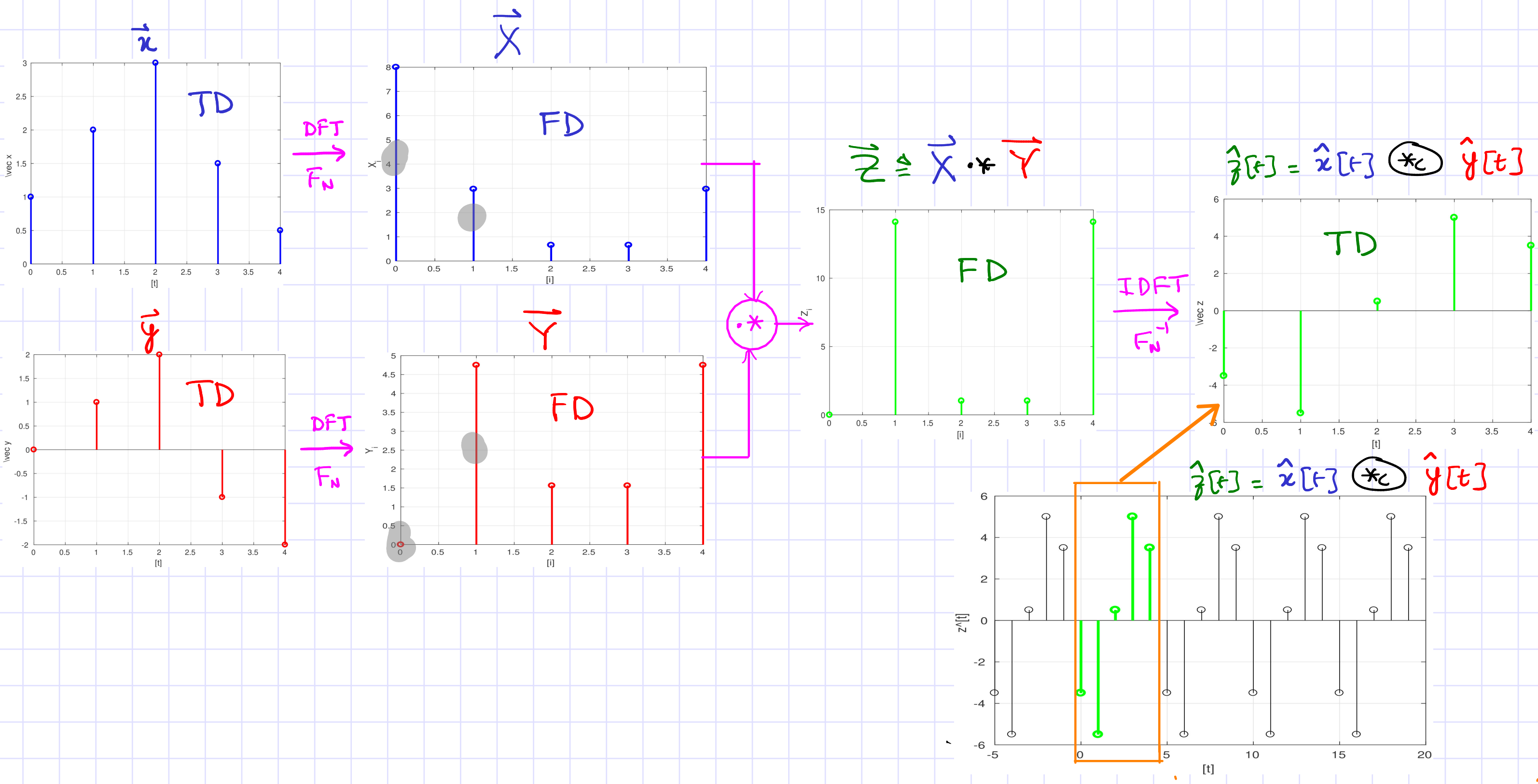
→ Define $\hat{z}[t] = \hat{x}[t] \circledast_c \hat{y}[t]$ ← CIRCULAR CONVOLUTION

→ Define $\vec{z} = [\hat{z}[0], \hat{z}[1], \ldots, \hat{z}[N-1]]^T$

→ Let $\vec{X} = F_N \vec{x}$, $\vec{Y} = F_N \vec{y}$, $\vec{Z} = F_N \vec{z}$ (i.e., take the DFTs of $\vec{x}, \vec{y}$, and $\vec{z}$)

$$\vec{X} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}, \quad \vec{Y} = \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{N-1} \end{bmatrix}, \quad \vec{Z} = \begin{bmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_{N-1} \end{bmatrix}$$

→ then: $\vec{Z} = \underbrace{\vec{X} .* \vec{Y}}_{\text{ELEMENT BY ELEMENT MULTIPLICATION}} = \begin{bmatrix} X_0Y_0 \\ X_1Y_1 \\ \vdots \\ X_{N-1}Y_{N-1} \end{bmatrix}$



→ DFT TURNS CIRCULAR CONVOLUTION IN THE TIME-DOMAIN INTO MULTIPLICATION IN THE FREQUENCY DOMAIN

THE DFT AND LTI SYSTEMS

→ RECALL: LTI SYSTEMS CHARACTERIZED BY IMPULSE RESPONSE $h[t]$

NOT CIRCULAR CONV.

→ $y[t] = u[t] \circledast h[t] = \sum_{i=0}^{t} u[i]\, h[t-i]$ (assuming $h[t]$ & $u[t]$ both causal)

→ Q: SUPPOSE I WANT TO CALCULATE $y[0], \cdots, y[n]$

→ HOW MANY MULTIPLICATION OPERATIONS DO I NEED ?

→ FOR $y[0]: 1$; $y[1]: 2$; $y[2]: 3$, $\cdots$, $y[n]: n+1$

$\Rightarrow 1+2+\cdots+(n+1) = \frac{(n+1)(n+2)}{2} \simeq \frac{n^2}{2}$ if n is large.

→ TRY $n \sim 2\times 10^6$ (1 minute of CD AUDIO) $\Rightarrow \frac{n^2}{2} \sim 2\times 10^{12} = 2$ TRILLION OPS

↳ ~10m COMPUTER TIME

→ CAN WE DO IT (MUCH) FASTER?

→ YES: USING THE DFT AND CIRCULAR CONVOLUTION

→ STRATEGY

→ FIRST, FIX THE NUMBER OF $y[\cdot]$ VALUES YOU WANT: $y[0], \cdots, y[N-1]$

DECIDE N

→ A: EXPRESS $y[t]$ AS A CIRCULAR CONVOLUTION, NOT JUST A REGULAR CONVOLUTION

→ $y[t] = u[t] \circledast h[t]$

$\overset{?}{=} \hat{u}_p[t] \circledast_c \hat{h}_p[t]$, $t = 0, \cdots, N-1$

→ CAN WE FIND SUITABLE $\hat{u}[t]$ and $\hat{h}_p[t]$?

→ B: IF WE CAN, THEN THE DFT CAN PERFORM CIRCULAR CONVOLUTION:

→ $\vec{Y}_p \triangleq (\text{DFT of } \vec{u}_p * (\text{DFT of } \vec{h}_p)$

→ $\vec{y}_p = \text{IDFT of } \vec{Y}_p$

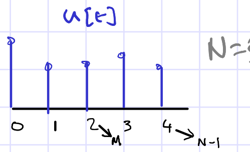
→ THE FIRST N entries SHOULD BE $y[0], \cdots, y[N-1]$

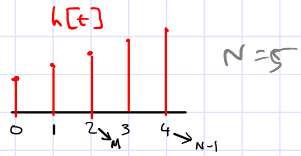
9D: EXPRESSING CONVOLUTION AS CIRCULAR CONVOLUTION

→ WE HAVE: $y[t] = \sum_{i=0}^{t} u[i]\,h[t-i]$ (u, h, BOTH CAUSAL)

→ WE ARE INTERESTED ONLY IN $t = 0, \cdots, N-1$

→ WILL NEED ONLY $u[0], \ldots, u[N-1]$ AND $h[0], \ldots, h[N-1]$





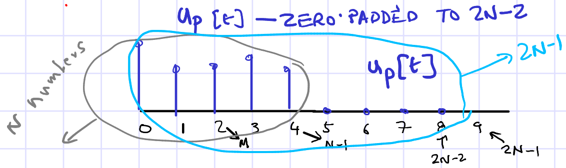
→ DEFINE $u_p[t]$ and $h_p[t]$: EXTENDED AND ZERO PADDED VERSIONS of $u[t]$ / $h[t]$

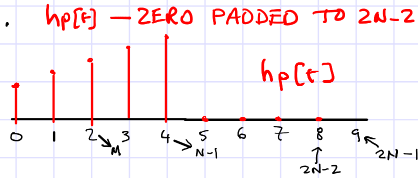
→ $u_p[t] = u[t]$, $t = 0, \cdots, N-1$ (N original TD values)

→ $u_p[N], u_p[N+1], \cdots, u_p[2N-2] = 0$ ⟵ (ADDITIONAL N−1 ZEROS)

→ $h_p[t] = h[t]$; $t = 0, \cdots, N-1$ (N original TD values)

→ $h_p[N], h_p[N+1], \cdots, h_p[2N-2] = 0$ ⟵ (ADDITIONAL N−1 ZEROS)

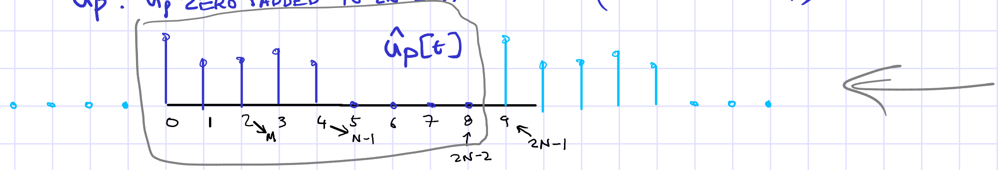




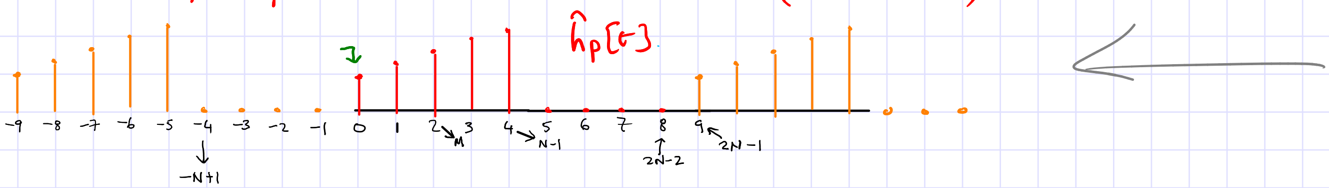
→ NOW EXTEND BOTH PERIODICALLY (MAKE THEM CIRCULAR)

$N \to 2N-1$

$\hat{u}_p$: u_p ZERO PADDED TO 2N−2 AND REPEATED (WITH PERIOD 2N−1)

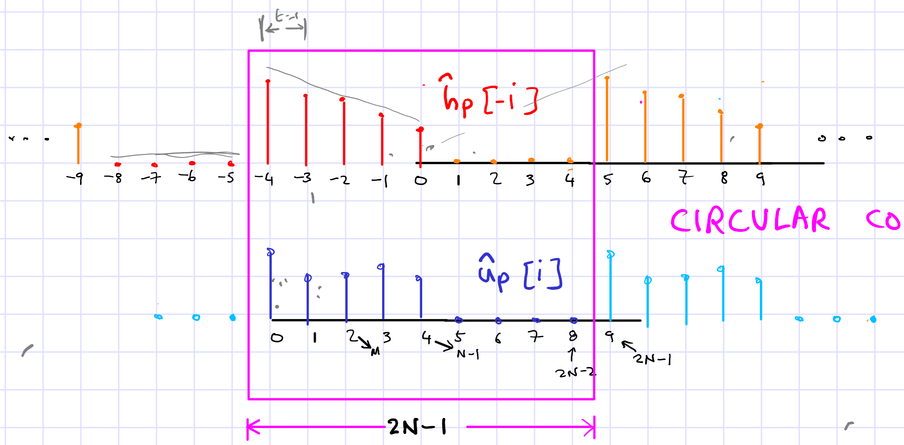


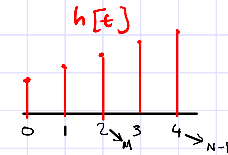
$\hat{h}_p$: h_p ZERO PADDED TO 2N−2 AND REPEATED (W PERIOD 2N−1)



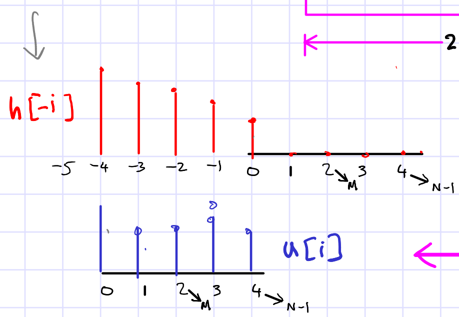
→ WHAT DOES THE CIRCULAR CONVOLUTION OF $\hat{u}_p$ and $\hat{h}_p$ LOOK LIKE?

→ vs REGULAR CONVOLUTION?





CIRCULAR CONVOLUTION OF $\hat{u}_p$ and $\hat{h}_p$



← REGULAR CONVOLUTION OF u[t] and h[t]

→ IDENTICAL FOR $t = 0, \dots, N-1$!

→ HENCE, CAN USE DFTs FOR REGULAR CONVOLUTION

9E. STRATEGY FOR FAST CONVOLUTION USING DFTs

→ CHOOSE $N = 2M+1$ (odd)

→ set $\vec{u}_p = \begin{cases} u[t], & t = 0, \dots, N-1 \\ 0, & t = N, \dots, 2N-2 \end{cases}$ $\quad \vec{u}_p \in \mathbb{R}^{2N-1}$

→ set $\vec{h}_p = \begin{cases} h[t], & t = 0, \dots, N-1 \\ 0, & t = N, \dots, 2N-2 \end{cases}$ $\quad \vec{h}_p \in \mathbb{R}^{2N-1}$

→ RUN 2 DFTs: $\vec{U}_p \triangleq F_{2N-1}\, \vec{x}_p$ ✓, $\vec{H}_p \triangleq F_{2N-1}\, \vec{x}_p$ ✓

→ SET $\vec{Y}_p = \vec{X}_p \; .\!* \; \vec{H}_p$ ✓

↳ ELEM. BY ELEM. MULT.

→ RUN an INVERSE DFT : $\vec{y}_p = F_{2N-1}^{-1}\, \vec{Y}_p$

→ $y[0] = y_p[0], \dots, y[N-1] = y_p[N-1]$ $(= u[t] \circledast h[t])$

→ IGNORE $y_p[N], \dots, y_p[2N-2]$

N log(N)

COMPUTATION

FFT: $\sim 2N \log(2N) \times 2$

$.*$: $\sim 2N$ MULTIPLICATIONS

IFFT: $\sim 2N \log(2N)$

→ $6N \left(\log(2N)\right) + 2N$

→ vs. $\frac{N^2}{2}$

→ TRY $N = 2 \times 10^6$

→ FFT BASED : 2.67×10^8

→ DIRECT : 2×10^{12}

$\rightarrow$ APPENDIX:

$$\vec{Z} = F_N \vec{z}$$

$$Z_i = \sum_{j=0}^{N-1} z_j\, \omega_N^{-ij}$$

— DFT TURNS CIRCULAR CONVOLUTION INTO MULTIPLICATION : PROOF

$\rightarrow$ GIVEN $z[t] = x[t] \circledast_t y[t] = \sum_{i=0}^{N-1} x[i]\, y[t-i]$

$$\rightarrow Z_i = \sum_{j=0}^{N-1} \omega_N^{-ij}\, z[j] = \sum_{j=0}^{N-1} \omega_N^{-ij} \sum_{k=0}^{N-1} x[k]\, y[j-k] = \sum_{k=0}^{N-1} x[k] \left(\sum_{j=0}^{N-1} \omega_N^{-ij}\, y[j-k] \right)$$

$$= \sum_{k=0}^{N-1} x[k] \Bigg[\underbrace{\sum_{j=0}^{k-1} \omega_N^{-ij}\, y[j-k]}_{\text{CALL THIS } A_k} + \underbrace{\sum_{j=k}^{N-1} \omega_N^{-ij}\, y[j-k]}_{\text{CALL THIS } B_k} \Bigg] = \sum_{k=0}^{N-1} x_k (A_k + B_k)$$

$$\longrightarrow A_k = \sum_{j=0}^{k-1} \omega_N^{-ij}\, y[j-k]$$

$\longrightarrow$ From circularity : $y[j-k] = y[j-k+N]$

$$\Rightarrow A_k = \sum_{j=0}^{k-1} \omega_N^{-ij}\, y[j-k] = \sum_{j=0}^{k-1} \omega_N^{-ij}\, y[j-k+N]$$

$\longrightarrow$ Define $l = j-k+N \Rightarrow j = l+k-N$

$$\Rightarrow A_k = \sum_{j=0}^{k-1} \omega_N^{-ij}\, y[j-k+N] = \sum_{l=N-k}^{N-1} \omega_N^{-i(l+k-N)}\, y[l] = \omega_N^{-ik} \sum_{l=N-k}^{N-1} \omega_N^{-il}\, y[l]$$

$$\longrightarrow B_k = \sum_{j=k}^{N-1} \omega_N^{-ij}\, y[j-k]$$

$\longrightarrow$ Let $l = j-k \Rightarrow j = l+k$

$$\Rightarrow B_k = \sum_{j=k}^{N-1} \omega_N^{-ij}\, y[j-k] = \sum_{l=0}^{N-k-1} \omega_N^{-i(l+k)}\, y[l] = \omega_N^{-ik} \sum_{l=0}^{N-k-1} \omega_N^{-il}\, y[l]$$

$$\Rightarrow A_k + B_k = \omega_N^{-ik} \sum_{l=N-k}^{N-1} \omega_N^{-il}\, y[l] + \omega_N^{-ik} \sum_{l=0}^{N-k-1} \omega_N^{-il}\, y[l] = \omega_N^{-ik} \underbrace{\sum_{l=0}^{N-1} \omega_N^{-il}\, y[l]}_{Y_i} = \omega_N^{-ik} Y_i$$

$$\Rightarrow Z_i = \sum_{k=0}^{N-1} (A_k + B_k)\, x_k = \sum_{k=0}^{N-1} \omega_N^{-ik} x_k Y_i = Y_i \underbrace{\sum_{k=0}^{N-1} \omega_N^{-ik} x_k}_{X_i} = X_i Y_i \longleftarrow \text{PROVED.}$$

→ ALGEBRAIC PROOF THAT $\hat{u}[t] \circledast_c \hat{h}[t] = u[t] \circledast h[t]$ for $t = 0, \dots, N-1$

$$\rightarrow \hat{u}[t] \circledast_c \hat{h}[t] = \sum_{i=0}^{2N-2} \hat{u}[i]\,\hat{h}[t-i]$$

$$\Rightarrow \hat{u}[t] \circledast_c \hat{h}[t] = \sum_{i=0}^{N-1} \hat{u}[i]\,\hat{h}[t-i] + \sum_{i=N}^{2N-2} \underbrace{\hat{u}[i]}_{0 \rightarrow \text{by padding}}\,\hat{h}[t-i]$$

$$= \sum_{i=0}^{N-1} \hat{u}[i]\,\hat{h}[t-i]$$

→ WE ARE INTERESTED ONLY IN $t = 0, \dots, N-1$

NOTE: $t-i$ ranges from $(-N+1)$ to $(N-1)$

NOTE 2: $\hat{h}[-N+1], \hat{h}[-N+2], \dots, \hat{h}[-1] = 0$

→ BECAUSE OF CIRCULARITY and ZERO PADDING:

$$\rightarrow \text{eg: } \hat{h}[-N+1] = \hat{h}[\underbrace{-N+1+2N-1}_{\text{CIRCULARITY}}] = \hat{h}[\underbrace{N}_{\text{FIRST ZERO PADDING LOCATION}}] = 0$$

$$\rightarrow \text{eg: } \hat{h}[-1] = \hat{h}[-1+2N-1] = \hat{h}[\underbrace{2N-2}_{\text{LAST ZERO PADDING LOCATION}}] = 0$$

$$\Rightarrow \hat{u}[t] \circledast_c \hat{h}[t] = \sum_{i=0}^{N-1} \hat{u}[i]\,\underbrace{\hat{h}[t-i]}_{0 \text{ if } i>t} = \sum_{i=0}^{t} \hat{u}[i]\,\hat{h}[t-i], \quad t = 0, \dots, N-1$$

→ t ranges from 0 to $N-1$

→ i ranges from 0 to $t \Rightarrow t-i$ ranges from 0 to t

→ i and $t-i$ both range from $0 \dots N-1$

$$\Rightarrow \hat{u}[i] = u[i] \text{ and } \hat{h}[t-i] = h[t-i]$$

$$\Rightarrow \hat{u}[t] \circledast_c \hat{h}[t] = \sum_{i=0}^{t} u[i]\,h[t-i] = u[t] \circledast h[t], \quad t = 0, \dots, N-1$$

→ SUMMARY:

→ ZERO-PAD $u[t]$ and $h[t]$ FROM $t=N$ to $t=2N-2$, THEN MAKE CIRCULAR/PERIODIC WITH PERIOD $2N-1$: CALL THESE $\hat{u}[t]$ and $\hat{h}[t]$

$$\Rightarrow \text{then } \hat{u}[t] \circledast_c \hat{h}[t] = u[t] \circledast h[t] \text{ for } t = 0, \dots, N$$