THE DISCRETE FOURIER TRANSFORM (DFT) AND ITS USES

1. INTRODUCTION AND PRELIMINARIES
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1. DFTs: INTRODUCTION

→ THE DFT IS JUST A MATRIX (SQUARE MATRIX)

→ BUT A VERY SPECIAL MATRIX

→ FILLED WITH (SPECIAL) COMPLEX NUMBERS
→ INVERTIBLE (AND ITS INVERSE IS "ALMOST ITSELF")

→ WIDELY USEFUL

→ CONVERT "TIME-DOMAIN" TO "FREQUENCY DOMAIN" AND VICE-VERSA
→ DIGITAL SIGNAL PROCESSING
→ FIND FREQUENCY SPECTRUM
→ PERFORM FILTERING
→ FOURIER ANALYSIS / SPECTROGRAMS / ETC.
→ IMAGE COMPRESSION
→ JPEG IMAGES (DCT)
→ FAST COMPUTATION: FFT

→ SHOW LIVE DEMO + SLIDES

→ CAN BE USED FOR A SPECIAL KIND OF INTERPOLATION (BAND-LIMITED INTERPOLATION)

→ SIMILAR TO SINC( ) INTERPOLATION

→ CONNECTIONS WITH GLOBAL POLYNOMIAL INTERPOLATION AND VANDERMONDE MATRIX

→ PRELIMINARIES:

→ COMPLEX CONJUGATE NOTATION: BAR OVER NUMBER/VECTOR/MATRIX: $\bar{a}$, $\bar{x}$, $\bar{A}$

→ HERMITIAN: MEANS COMPLEX CONJUGATE TRANSPOSE

→ DENOTED BY $*$ or H SUPERSCRIPT

→ $1^* = 1$; $(1+j)^* = (1-j)$

→ $$\begin{bmatrix} 1 & e^{j\pi} \\ 2 & 1+j \end{bmatrix}^* = \begin{bmatrix} 1 & 2 \\ e^{-j\pi} & 1-j \end{bmatrix}$$

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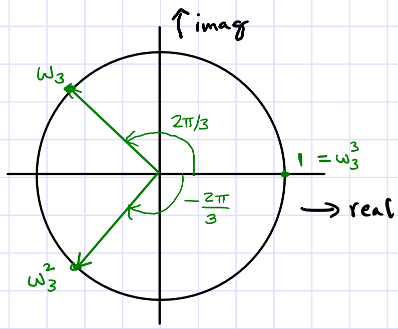
2. ROOTS OF 1 ON THE COMPLEX PLANE

→ $\sqrt{1} = \pm 1$, FOCUS ON THE LESS OBVIOUS ONE: -1.

→ $\sqrt[3]{1} = 1,\ e^{+j\frac{2\pi}{3}},\ e^{-j2\pi/3}$



$$\left(e^{j2\pi/3}\right)^2 = e^{j\frac{2\pi}{3}\cdot 2}$$

$$\left(e^{j\frac{2\pi}{3}}\right)^3 = e^{j2\pi} = 1.$$

— FOCUS ON THE ONE w/ +ve angle:

→ $\omega_3 \triangleq e^{j2\pi/3}$

→ $\omega_3^3 = 1$

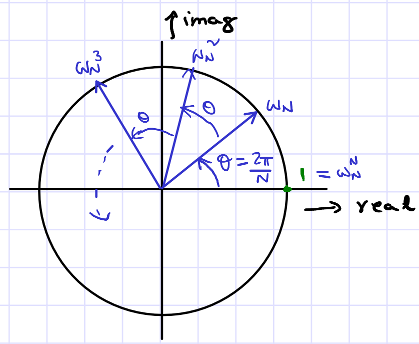
→ $\sqrt[N]{1} =$

small

"omega N"

→ $\omega_N \triangleq e^{j\frac{2\pi}{N}}$

→ $\omega_N^N = 1$



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3. THE DFT MATRIX

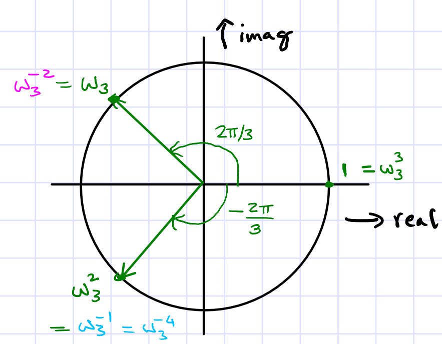
→ CONVENTION: ROW/COL INDICES i, j GO FROM $0, \ldots, N-1$

→ DFT MATRIX F_N : $(i,j)^{th}$ entry $= \omega_N^{-ij}$

→ EXAMPLE: $N=3$

$$F_3 = \begin{array}{c|c} i \downarrow \; j \rightarrow & \begin{matrix} 0 & \quad 1 & \quad 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} \omega_3^{-0\times 0} & \omega_3^{-0\times 1} & \omega_3^{-0\times 2} \\ \omega_3^{-1\times 0} & \omega_3^{-1\times 1} & \omega_3^{-1\times 2} \\ \omega_3^{-2\times 0} & \omega_3^{-2\times 1} & \omega_3^{-2\times 2} \end{bmatrix} \end{array} = \begin{array}{c|c} & \begin{matrix} 0 & \quad 1 & \quad 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3^{-1} & \omega_3^{-2} \\ 1 & \omega_3^{-2} & \omega_3^{-4} \end{bmatrix} \end{array} \updownarrow N=3$$

$\longleftarrow N=3 \longrightarrow$



→ DFT - VECTOR PRODUCT

→ say $\vec{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \triangleq F_N \vec{x}$

→ Then: $y_i = \sum_{j=0}^{N-1} \omega_N^{-ij} x_j$, $i = 0, \dots, N-1$ (WILL BE USEFUL LATER)

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4. PROPERTIES OF THE DFT

A. SYMMETRIC: $F_N^T = F_N$ $\quad$ ($(i,j)^{th}$ entry is ω_N^{-ij})

B. FULL RANK ≡ INVERTIBLE

→ $F_N^{-1} = \frac{1}{N} F_N^{*}$ ← HERMITIAN $\quad$ ← INVERSE DFT ≡ IDFT

→ OR: $F_N F_N^{*} = N I_N$ (ALMOST UNITARY)

→ PROOF:

→ LET $A = F_N F_N^{*}$ ← WILL SHOW THIS IS $N I_N$

→ WHAT IS a_{ij}, the $(i,j)^{th}$ entry of A?

→ $a_{ij} = (i^{th}$ row of $F_N) \times (j^{th}$ col of $F_N^{*})$

$$= \sum_{k=0}^{N-1} \omega_N^{-ik} \omega_N^{+jk} = \sum_{k=0}^{N-1} \omega_N^{k(j-i)} = \underbrace{\left[\omega_N^{(j-i)}\right]}_{\text{call this } p_{ij}}{}^{0} + \left[\omega_N^{(j-i)}\right]^{1} + \dots + \left[\omega_N^{(j-i)}\right]^{N-1}$$

$$= p_{ij}^0 + p_{ij}^1 + p_{ij}^2 + \dots + p_{ij}^{N-1}$$

→ IF $i=j$, then $p_{ij} = \omega_N^0 = 1$

$\Rightarrow a_{ii} = N$

→ IF $i \neq j$:

→ $a_{ij} = p_{ij}^0 + p_{ij}^1 + p_{ij}^2 + \dots + p_{ij}^{N-1}$

→ $p_{ij} a_{ij} = p_{ij}^1 + p_{ij}^2 + \dots + p_{ij}^{N-1} + p_{ij}^{N}$

→ SUBTRACT

$\Rightarrow a_{ij} - p_{ij} a_{ij} = p_{ij}^0 - p_{ij}^N \Rightarrow a_{ij}(1 - p_{ij}) = 1 - p_{ij}^N \Rightarrow a_{ij} = \frac{1 - p_{ij}^N}{1 - p_{ij}}$

→ $p_{ij}^N = \omega_N^{N(i-j)} = 1$

$\Rightarrow a_{ij} = \frac{1 - p_{ij}^N}{1 - p_{ij}} = 0$

→ SUMMARY: $a_{ij} = \begin{cases} N, & i = j \\ 0, & i \neq j \end{cases} \Rightarrow A = F_N F_N^{*} = N I_N$

→ Q: GIVEN THAT $N I_n = F_N F_N^*$, WHAT PROPERTY DO THE ROWS/COLS OF F_N HAVE?

→ A: THEY ARE <u>ORTHOGONAL</u> (BUT NOT <u>ORTHONORMAL</u>)

→ <u>ALSO</u> : THE SUM OF EVERY ROW/COL (EXCEPT THE FIRST) = 0
→ SUM OF FIRST ROW/COL = N.

→ PROVE THIS YOURSELVES

<u>C.</u> $IDFT = DFT^{-1} = F_N^{-1} = \frac{1}{N} F_N^*$

→ <u>IDFT – VECTOR PRODUCT</u>

→ say $\vec{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} \triangleq IDFT \times \vec{x} = \frac{1}{N} F_N^* \vec{x}$

→ Then: $y_i = \frac{1}{N} \sum_{j=0}^{N-1} W_N^{ij} x_j$, $i = 0, \cdots, N-1$ (WILL BE USEFUL LATER)

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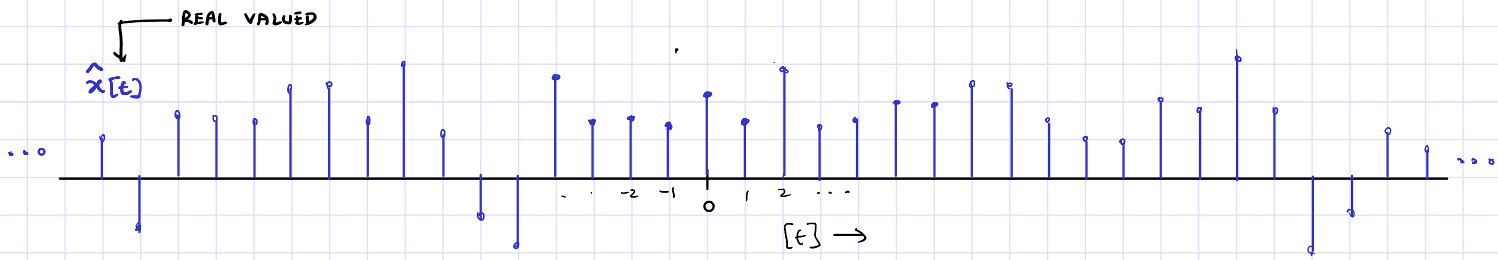
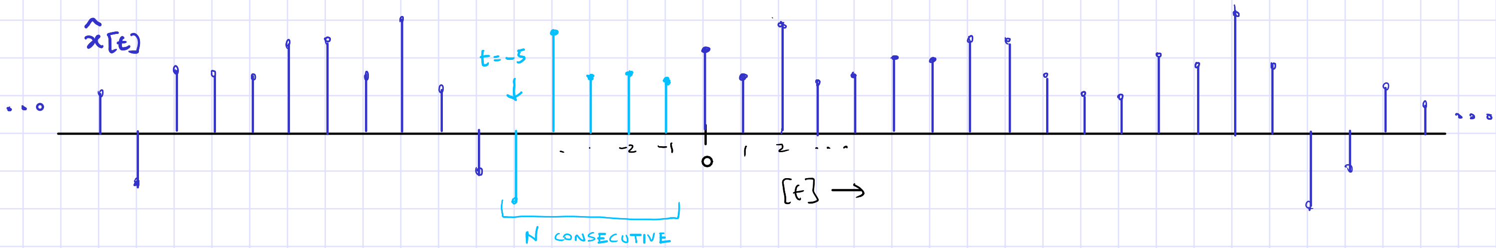
5. VECTORS FROM SEQUENCES

→ TURNING A CHUNK OF A SEQUENCE INTO A VECTOR
→ GIVEN SOME DISCRETE-TIME SEQUENCE : $\hat{x}[t]$, $t = -\infty, \cdots, -1, 0, 1, 2, \cdots \infty$



→ PICK ANY N CONSECUTIVE VALUES

→ EXAMPLE : $N = 5$, $t = -5, -4, \cdots, -2, -1$.



→ MAKE A SIZE-N VECTOR OF THEM: $\vec{x} = \begin{array}{cl} \begin{bmatrix} x[-5] \\ x[-4] \\ x[-3] \\ x[-2] \\ x[-1] \end{bmatrix} & \begin{array}{l} \leftarrow x_0 \\ \leftarrow x_1 \\ \vdots \\ \\ \leftarrow x_{N-1} \end{array} \end{array}$

→ WE CAN NOW RUN DFTs (OR OTHER VECTOR OPERATIONS) ON $\vec{x}$

→ THIS IS OFTEN USEFUL

→ E.G., THE SPECTROGRAM SHOWN EARLIER (DEMO/SLIDES) DOES JUST THIS, USING CONSECUTIVE CHUNKS OF THE INPUT EVERY TIME THE DISPLAY IS UPDATED.

→ UNLESS OTHERWISE SPECIFIED, WE WILL ASSUME BELOW THAT VECTORS CONTAIN CHUNKS FROM SEQUENCES, STARTING FROM $t=0$

→ e.g.:

$$\vec{u} = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix} \begin{array}{l} \leftarrow u_0 \\ \leftarrow u_1 \\ \vdots \\ \leftarrow u_{N-1} \end{array}$$

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6. DFTs OF REAL VECTORS: COMPLEX CONJUGACY PROPERTIES

→ GIVEN A SEQUENCE CHUNK / VECTOR $\vec{x}$ OF LENGTH N, ALL REAL NUMBERS

→ i.e., $\vec{x} \in \mathbb{R}^N$

small x

→ TAKE ITS DFT:

$$\vec{X} = F_N \, \vec{x}$$

CAPITAL X

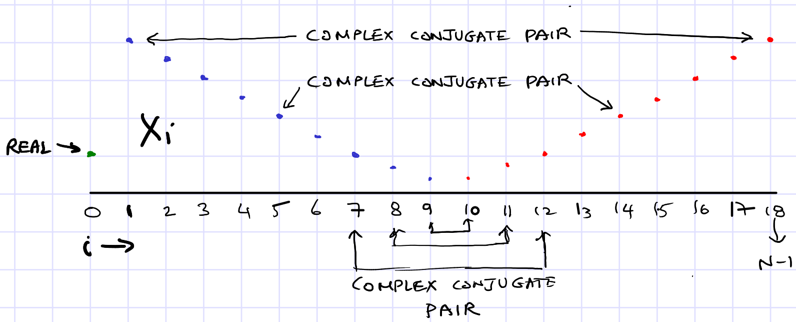
→ THEN THE MEMBERS OF $\vec{X}$ HAVE A COMPLEX CONJUGACY PROPERTY:

→ $X_{N-i} = \overline{X_i}$, $i = 1, \ldots, N-1$

NOTE: STARTING FROM 1 (not 0)

→ AND: X_0 IS REAL.

→ IN PICTURES:



→ IF N IS EVEN, i.e., $N = 2M$ FOR SOME NATURAL NUMBER M

→ THEN $N - M = 2M - M = M$

$\Rightarrow$ $\overline{X_{N-i}} = X_i$ becomes (with $i = M$) $\overline{X_M} = X_M$, i.e., X_M is REAL.

→ NOT TRUE IF N IS ODD

→ PROOF OF COMPLEX CONJUGACY PROPERTY

A) $X_0 = \sum_{j=0}^{N-1} \underbrace{\omega_N^{-0\cdot j}}_{1} x_i = \sum_{j=0}^{N-1} x_i$. THIS IS REAL SINCE x_i are all real.

B) $X_{N-i} = \sum_{j=0}^{N-1} \omega_N^{-(N-i)j} x_i = \sum_{j=0}^{N-1} \overset{1}{\omega_N^{-Nj}} \omega_N^{ij} x_i = \sum_{j=0}^{N-1} \overline{\left(\omega_N^{-ij}\right)} x_i$ ← COMPLEX CONJUGATE

$= \sum_{j=0}^{N-1} \overline{\left(\omega_N^{-ij}\right)}\, \overline{x_i} = \overline{\sum_{j=0}^{N-1} \left(\omega_N^{-ij}\right) x_i} = \overline{X_i}$

$\overline{x_i} = x_i$ BECAUSE x_i IS REAL

→ COROLLARY : SUPPOSE $X_{N-i} \neq \overline{X_i}$, THEN $\vec{X}$ IS NOT THE DFT OF A REAL VECTOR

→ A SMALL EXERCISE FOR YOU

→ GIVEN SOME $\vec{X} \in \mathbb{C}^N$ ← COMPLEX NUMBERS

→ OBEYS X_0 IS REAL AND $X_{N-i} = \overline{X_i}$ for $i = 1, \dots, N-1$

→ DEFINE $\vec{x}$ = IDFT of $\vec{X} = \frac{1}{N} F_N^* \vec{X}$

→ PROVE THAT $\vec{x}$ is REAL.

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7. FREQUENCY INTERPRETATION OF THE DFT

→ OUR GOAL HERE IS TO TRY TO DEVELOP A FEELING/INTUITION FOR WHAT THE ENTRIES OF $\vec{X}$ REPRESENT

→ WE WILL ASSUME $\vec{x}$ (and $x(t)$) are always real.

→ THE FLOW IS LIKE THIS :

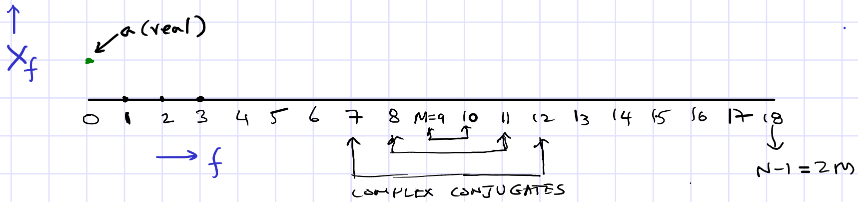
→ WE WILL MAKE UP SOME SIMPLE EXAMPLES OF $\vec{X}$ (consistent w/ the conjugacy conditions)

→ WE WILL IDFT $\vec{X}$ to get $\vec{x}$

→ LOOKING AT THE ENTRIES OF $\vec{x}$ WILL GIVE US INSIGHT.

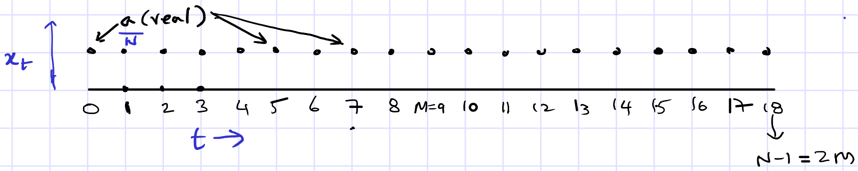
N = 2M

→ Choose $\underbrace{X_0 = a}_{}$, $X_f = 0$, $f = 1, 2, \cdots, N-1$ (REAL: a)



→ WHAT IS ITS IDFT?

$$\rightarrow \quad x_t = \frac{1}{N}\sum_{f=0}^{N-1} W_N^{tf} X_f = \frac{a}{N}, \quad \text{for } t = 0, 1, 2, \cdots, N-1$$



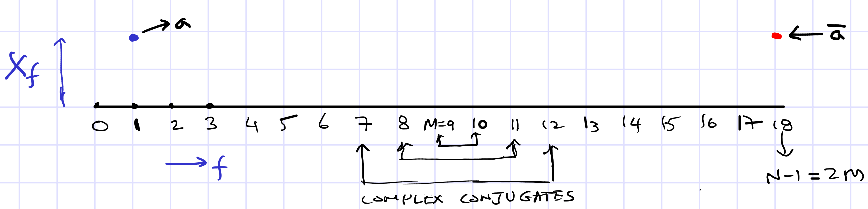
$x[t]$ constant with $t \in \{0, \cdots, N-1\}$

$\Longrightarrow$ DC

→ THEREFORE, X_0 CAPTURES A DC WAVEFORM (interpreting x_t as $x[t]$)

→ WHAT ABOUT X_1?

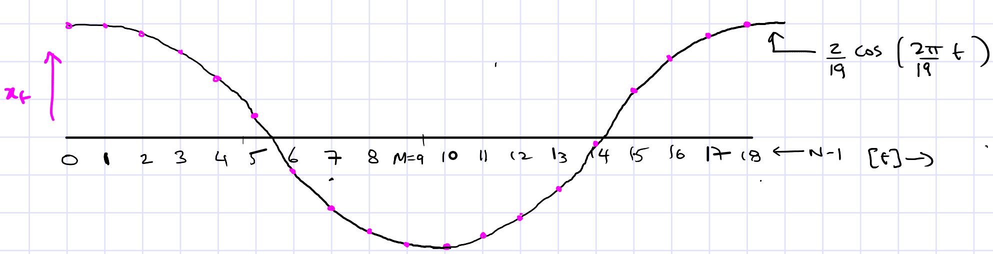
→ CHOOSE: $X_0 = 0$, $X_1 = a$ (COMPLEX), $X_{N-1} = \bar{a}$, the rest $= 0$



$$\rightarrow \quad x_t = \frac{1}{N}\sum_{f=0}^{N-1} W_N^{tf} X_f = \frac{1}{N}\left(W_N^{t} a + W_N^{t(N-1)} \bar{a}\right) = \frac{1}{N}\left(W_N^{t} a + W_N^{-t} \bar{a}\right)$$

$$= \frac{2}{N} \operatorname{Re}\{W_N^{t} a\}$$

→ say $a = 1$ (for simplicity), then $x_t = \frac{2}{N}\cos\left(\frac{2\pi}{N} t\right)$



→ THIS IS A (CO)SINE WAVE (DISCRETE SAMPLES OF IT)

→ SUPPOSE $X_1 = a$ (some general complex number), $X_{N-1} = \bar{a}$

→ consider $X_1 = a$ to be a PHASOR

→ and write it in magnitude-phase form: $a = \underset{\uparrow \text{ real}, >0}{A} e^{j\theta}$

→ then $x(t) = \frac{2}{N} \mathrm{Re}\{ \omega_N^t a \} = \frac{2}{N} \mathrm{Re}\{ \omega_N^t A e^{j\theta} \}$

$$= \frac{2A}{N} \mathrm{Re}\{ e^{j\frac{2\pi t}{N}} e^{j\theta} \} = \frac{2A}{N} \mathrm{Re}\{ e^{j(\frac{2\pi}{N}t + \theta)} \}$$

$$\Rightarrow x_t = \underbrace{\frac{2A}{N}}_{\text{Amplitude}} \cos(\underbrace{\frac{2\pi}{N}}_{\text{frequency}} t + \underbrace{\theta}_{\text{phase shift.}})$$

→ compare against what a "regular phasor" means:

→ Given $a = Ae^{j\theta}$, the time-domain waveform is:

$$a e^{j\omega t} + \bar{a} e^{-j\omega t} = 2 \mathrm{Re}\{ a e^{j\omega t} \} = 2A \cos(\omega t + \theta)$$

→ SO NOW WE SEE THAT IF we interpret X_1 as a phasor, x_t is simply the samples of the regular phasor's time-domain waveform:

→ scaled by $1/N$

→ with $\omega = \frac{2\pi}{N}$ ← THIS IS CALLED THE FUNDAMENTAL (ANGULAR) FREQUENCY

→ WHAT ABOUT $X_2, X_3, \cdots$?

→ LET'S BE A BIT MORE PRECISE.

→ TAKE N TO BE ODD $\Rightarrow N = 2\underset{\uparrow \text{ INTEGER} > 0}{M} + 1$

→ TAKE $X_i = a$, $X_{N-i} = \bar{a}$, for some $0 < i \leq M$

→ with $a = Ae^{j\theta}$, as before.

→ THEN, $x_t = \frac{1}{N} \sum_{f=0}^{N-1} \omega_N^{ft} X_f = \frac{1}{N}(\omega_N^{it} X_i + \omega_N^{(N-i)t} X_{N-i}) = \frac{1}{N}(\omega_N^{it} a + \omega_N^{-it} \bar{a})$

$$= \frac{2}{N} \mathrm{Re}\{ \omega_N^{it} a \} = \frac{2A}{N} \cos\left(\frac{2\pi i}{N} t + \theta \right)$$

→ IF WE INTERPRET X_i as a phasor, x_t = samples of the "regular phasor time domain waveform",

→ scaled by $\frac{1}{N}$

→ with $\omega = \frac{2\pi}{N} i$ ← i^{th} HARMONIC OF FUNDAMENTAL $= i \, (\underset{\uparrow \text{ FUNDAMENTAL}}{\tfrac{2\pi}{N}})$

→ <u>SUMMARY</u>

→ take N odd ⇒ $N = 2M+1$

→ $X_0 = a$ (real) $\longmapsto$ DC $\vec{x} = \frac{a}{N}$

→ $X_i = a$ (phasor) $\longmapsto$ sinusoidal $\vec{x}$: $x_t = \frac{2}{N}|a| \cos\left(\frac{2\pi i}{N} t + \measuredangle a\right)$ → angle of a, ie, θ

→ for $i = 1, \cdots, M$

→ $X_{N-i} = \bar{a}$ REQUIRED FOR $\vec{x}$ to be REAL.

→ POINT TO PONDER (AND WORK OUT ON YOUR OWN)

→ HOW DOES THE ABOVE CHANGE IF N IS <u>EVEN</u>?

→ <u>NOW WE SEE WHAT THE DFT DOES:</u>

→ $\vec{X} = F_N \vec{x} \iff \vec{x} = \frac{1}{N} F_N^* \vec{X}$

→ $\vec{X}$ CONTAINS:

→ the DC component (X_0)

→ phasors with angular frequencies $\frac{2\pi i}{N}$, $i = 1, \cdots, M$

→ <u>I.E.</u>, $\vec{x}$ ($= x[t]$) HAS BEEN DECOMPOSED BY THE DFT INTO A SUM OF $\underbrace{\text{THE ABOVE PHASORS WITH THE ABOVE FREQUENCIES}}$ + <u>DC</u>.

↑ DISCRETE SAMPLES OF SINUSOIDS WITH FREQUENCIES $\frac{2\pi i}{N}$ ← FUNDAMENTAL, $i=1,\ldots,M$. AND HARMONICS.

→ THIS IS WHY $\vec{X}$ is CALLED THE FREQUENCY-DOMAIN REPRESENTATION OF $\vec{x}$.

→ CLAIM: THE DFT LOCATES THE FREQUENCY OF A TIME DOMAIN SINUSOID

→ MORE PRECISELY, THE CLAIM IS:

→ Let $M > 0$, and $N = 2M+1$

→ SUPPOSE $x(t) = A_k \cos\left(\frac{2\pi}{N} kt + \theta_k\right)$, where k is ONE of $1, \dots, M$

→ TAKE SAMPLES OF $x(t)$ at $t = 0, \dots, N-1$: this is $\vec{x} \in \mathbb{R}^N$.

→ TAKE THE DFT of $\vec{x}$: $\vec{X} = F_N \vec{x}$

→ THEN THE ENTRIES OF $\vec{X}$ ARE AS FOLLOWS:

→ $X_k = \frac{N}{2} A_k e^{j\theta_k}$, $X_{N-k} = \overline{X_k}$, all other $X_i = 0$

PROOF:

$$\to X_i = \sum_{l=0}^{N-1} W_N^{-il} x_l \quad , \quad i = 0, \dots, N-1 \qquad (1)$$

$$\to x_l = A_k \cos\left(\frac{2\pi}{N} kl + \theta_i\right) = \frac{A_k}{2} e^{j\theta_i} e^{j\frac{2\pi}{N}kl} + \frac{A_k}{2} e^{-j\theta_i} e^{-j\frac{2\pi}{N}kl}$$

$$= \frac{A_k}{2} e^{j\theta_i} W_N^{kl} + \frac{A_k}{2} e^{-j\theta_i} W_N^{-kl} \qquad (2)$$

→ Putting (2) in (1):

$$\to X_i = \sum_{l=0}^{N-1} W_N^{-il} \left[\frac{A_k}{2} e^{j\theta_i} W_N^{kl} + \frac{A_k}{2} e^{-j\theta_i} W_N^{-kl} \right]$$

$$= \frac{A_k}{2} e^{j\theta_i} \sum_{l=0}^{N-1} W_N^{(k-i)l} + \frac{A_k}{2} e^{-j\theta_i} \sum_{l=0}^{N-1} W_N^{-(i+k)l}$$

$$= \frac{A_k}{2} e^{j\theta_i} \sum_{l=0}^{N-1} W_N^{(k-i)l} + \frac{A_k}{2} e^{-j\theta_i} \sum_{l=0}^{N-1} W_N^{-(i+k)l} \overset{1}{W_N^N}$$

$$= \frac{A_k}{2} e^{j\theta_i} \underbrace{\sum_{l=0}^{N-1} W_N^{(k-i)l}}_{\text{call this } C} + \frac{A_k}{2} e^{-j\theta_i} \underbrace{\sum_{l=0}^{N-1} W_N^{-(i+k-N)l}}_{\text{call this } D} \qquad (3)$$

→ BOTH C and D are of the form: $\sum_{l=0}^{N-1} W_N^{pl}$ (4)

→ FOR C, $p=(k-i)$

→ with $k \in \{1,\dots,M\}$ and $i \in \{0,\dots,N-1\}$, we have $p \in \{-(N-2),\dots,M\}$ (5)

→ FOR D, $p=(N-k-i)$

→ with $k \in \{1,\dots,M\}$ and $i \in \{0,\dots,N-1\}$, we have $p \in \{-(M-1),\dots,(N-1)\}$ (6)

→ (4) is A GEOMETRIC SERIES SUM:

$$\rightarrow S_p \triangleq \sum_{l=0}^{N-1} W_N^{pl} = 1 + W_N^{p} + W_N^{2p} + \dots + W_N^{(N-1)p} \quad (7)$$

$$\Rightarrow W_N^p S_p = W_N^p + \dots + W_N^{(N-1)p} + W_N^{Np}$$

→ SUBTRACT

$$\rightarrow S_p(1-W_N^p) = 1 - W_N^{Np}$$

$$\Rightarrow S_p = \frac{1-W_N^{Np}}{1-W_N^p} = \begin{cases} N, & \text{if } p=0 \\ 0, & \text{if } p \neq 0 \text{ and } |p|<N \end{cases} \quad (8)$$

→ EXPLANATION:

→ if $p=0$, then use (4) directly to get $S=N$

→ you can also get this using L'Hospital's Rule on $\frac{1-W_N^{Np}}{1-W_N^p}$

→ if $p \neq 0$ and $|p|<N$, then $W_N^p \neq 1$

→ hence $1-W_N^p \neq 0$ (i.e., the denominator is $\neq 0$)

→ $W_N^{Np} = (W_N^N)^p = 1 \Rightarrow 1 - W_N^{Np} = 0$ (i.e., the numerator is 0)

→ USING (8), (5) and (6), (3) becomes

$$\rightarrow X_i = \frac{A_k}{2} e^{j\theta_i} S_{k-i} + \frac{A_k}{2} e^{-j\theta_i} S_{N-k-i} \quad (9)$$

→ THEREFORE, X_i IS NONZERO ONLY IF $k-i=0$ OR $N-k-i=0$:

$$\rightarrow X_k = N \frac{A_k}{2} e^{j\theta_i}$$

$$\rightarrow X_{N-k} = N \frac{A_k}{2} e^{-j\theta_i} = \overline{X_k}$$

→ $X_i = 0$ if $i \neq k$ or $N-k$

→ PROVED